3 Spanning sets, linear independence, and bases

We have defined many vector spaces, but we started by looking at $\mathbb{R}^n$, which is much easier to think about. One of the nicest and most helpful things about $\mathbb{R}^n$ is the existence of *coordinates*. Rather than, say, just drawing a point on a graph, or perhaps giving an angle and a distance, we can specify a point in $\mathbb{R}^3$ by giving its x-coordinate, its y-coordinate, and its z-coordinate. And similarly, we can specify a point in $\mathbb{R}^7$ by specifying seven real-number coordinates.

In contrast, it's not really clear what it means to talk about coordinates for $\mathcal{F}(\mathbb{R}, \mathbb{R})$. But if we had coordinates there, it would make our life much easier. (In particular, physicists often want to talk about subspaces of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ and then put coordinates on them and treat them like $\mathbb{R}^n$). So we would like to find a way to put coordinates on any vector space V.

We'll see that any "coordinate system" will need to have two basic properties: first, we want it to represent any vector in our vector space; second, we want it to represent each vector only once. We will treat these two criteria separately, and then show that we can always find a set that has both properties, which we will call a "basis".

3.1 Spanning sets

Definition 3.1. If V is a vector space $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a list of vectors in V, then a *linear combination* of of the vectors in S is a vector of the form

$$\sum_{i=1}^{n} a_i \mathbf{v}_i = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_n \mathbf{v}_n$$

where $a_i \in \mathbb{R}$ are (real number) scalars.

A linear combination of vectors in V will always itself be an element of V, since V is closed under scalar multiplication and under vector addition.

Geometrically, a linear combination of vectors represents some destination you can reach only going in the directions of your chosen vectors (for any distance. So if I can go north or west, any distance"northwest" will be a linear combination of those vectors. And "southeast" will as well, since we can always go in the "opposite" direction. But "up" will not be.

Remark 3.2. This is a "linear" combination because it combines the vectors in the same way a line or plane does—adding all the vectors together, but with some coefficient. We will revisit this terminology in the next section when we discuss linear functions.

It's totally possible to have a linear combination of infinitely many vectors. But studying these requires some sense of convergence, and thus calculus/analysis. So we won't talk about it in *this* class, except for the occasional aside.

Example 3.3. Let $V = \mathbb{R}^3$ and let $S = \{(1, 0, 0), (0, 1, 0)\}$. Then we see that

$$(3, 2, 0) = 3(1, 0, 0) + 2(0, 1, 0) \quad \text{and} \quad (-5, 3\pi, 0) = -5(1, 0, 0) + 3\pi(0, 1, 0)$$

are linear combinations of vectors in S.

However, $(1, 1, 1)$ is *not* a linear combination of vectors in S. If it were, we would have

$$a(1, 0, 0) + b(0, 1, 0) = (1, 1, 1)$$

and thus $(a, b, 0) = (1, 1, 1)$ which cannot happen for any $a, b \in \mathbb{R}$.

We see that this idea can tell us how coordinates work: a set of coordinates for V is a set of vectors S where we can build any vector in V as a linear combination of vectors in S. So the next natural question is to take a set S and ask what vectors we can get by taking linear combinations of vectors in S.

Definition 3.4. Let V be a vector space $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of vectors in V. We say the *span* of S is the set of all linear combinations of vectors in S, and write it $\operatorname{Span}(S)$ or $\operatorname{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$.

For notational consistency, we define the span of the empty set $\operatorname{Span}(\{\})$ to be the trivial vector space $\mathbf{0} = \{\mathbf{0}\}$.

Example 3.5. As before, take $V = \mathbb{R}^3$ and $S = \{(1, 0, 0), (0, 1, 0)\}$. Then

$$\operatorname{Span}(S) = \{a(1, 0, 0) + b(0, 1, 0)\} = \{(a, b, 0)\}.$$

Now let $T = \{(3, 2, 0), (13, 7, 0)\}$. Then

$$\operatorname{Span}(T) = \{a(3, 2, 0) + b(13, 7, 0)\} = \{(3a + 13b, 2a + 7b, 0)\}.$$

Notice that these are actually the same set! The first spanning set "looks" nicer, but it's hard to make this sense of "nice" mathematically precise. We'll do our best, but won't really get there until section 6.

However, we *can* tell whether two sets of vectors have the same span pretty easily. This requires us to define a new term:

Definition 3.6. If $A = (a_{ij})$ is a $m \times n$ matrix, then each row can be viewed as a vector in $\mathbb{R}^n$; we call these vectors the *row vectors* of A. We may notate them as $\mathbf{r}_i = (a_{i1}, a_{i2}, \dots, a_{in})$. The span of the row vectors of A is the *row space* of A.

Now we can use what we learned about matrices in section 1.3 to figure out the span of a set of vectors.

Proposition 3.7. *Two row-equivalent matrices have the same row space.*

Proof. We need to check that each elementary row operation doesn't change the span of the set of vectors.

I. (Switch two rows) Switching the order of two vectors does not affect the span at all.

II. (Multiply a row by a nonzero scalar) Multiplying a vector by a non-zero scalar won't change the span of the set of vectors, since in any linear combination we can always just multiply the relevant coefficient by the inverse of our non-zero scalar.

III. (Add a multiple of one row to another) This won't add anything to the span, since a linear combination of the new vectors will still be a linear combination of the old vectors.

This won't lose anything from the span, since we can undo the row operation, and so every old vector is a linear combination of new vectors.

□

Example 3.8. $\operatorname{Span}(T) = \operatorname{Span}(\{(3,2,0),(13,7,0)\})$ is the rowspace of

$$\begin{bmatrix} 3 & 2 & 0 \\ 13 & 7 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & 0 \\ 39 & 21 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & 0 \\ 0 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus we see that $\operatorname{Span}(\{(3,2,0),(13,7,0)\}) = \operatorname{Span}(\{(1,0,0),(0,1,0)\})$.

Example 3.9. Take $V = \mathbb{R}^3$ and let $U = \{(1,0,0),(0,1,0),(1,1,0)\}$. What is $\operatorname{Span}(U)$?

We see that

$$\operatorname{Span}(U) = \{a(1,0,0) + b(0,1,0) + c(1,1,0)\} = \{(a+c, b+c, 0)\}.$$

In this case, it's not too hard to see that this is the same as the set $\operatorname{Span}(\{(1,0,0),(0,1,0)\}) = \{(a,b,0)\}$. But we can also use row reduction:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have now described the set $\{(a, b, 0)\}$ as the span of three different sets. Two of these sets have had two elements, and one has had three. It's not too difficult to describe it as the span of a set that's as large as we want: for instance, if we take $S = \{(a, b, 0)\}$ to be the set of all vectors with third coordinate zero, then a little thought will tell us that $\text{Span}(S) = S$. At the other extreme, it turns out we need to start with at least two elements to span all of S; we will prove this in section 3.2.

We've discussed the idea of spanning algebraically; what is happening geometrically? Recall that each vector gives us a direction and a distance. Since we can multiply our vectors by any scalar, that means we can go any distance in that direction. And since we can add vectors together, that means we can go in one direction, and then another direction. So the span of a set is all of the places I can get to by only going in the direction of vectors in that set.

Let's return to our first example. Our set S included the vectors $(1, 0, 0)$ and $(0, 1, 0)$; those correspond to "north" and "east" in Euclidean threespace. So the span of S is the set of all locations I can get to by going some distance north and then some distance east. But neither of these vectors moves me at all up and down, so I cannot change my height.

In our third example, we had the vectors $(1, 0, 0), (0, 1, 0), (1, 1, 0)$. So we can go north, or east, or north-east. But north-east doesn't open any new options since we could already go north and then east. And we still can't change our height.

Example 3.10. Let $S = \{(1, 2, 3, 4), (1, 1, 1, 1)\}$. Is $(0, 0, 2, 2)$ in $\text{Span}(S)$? Is $(0, 1, 2, 3)$?

Each question like this is really asking us to solve a system of linear equations, and thus we can easily solve it using row reduction.

If $(0, 0, 2, 2) \in \text{Span}(S)$ then we can write $(0, 0, 2, 2) = a(1, 2, 3, 4) + b(1, 1, 1, 1)$. Then we have the system

$$\begin{aligned} a + b &= 0 \\ 2a + b &= 0 \\ 3a + b &= 2 \\ 4a + b &= 2 \end{aligned}$$

which gives us the matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 2 \\ 4 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 0 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{array}\right].$$

This system isn't solvable, since the last two equations are $0 = 2$. Thus $(0, 0, 2, 2) \notin \mathrm{Span}(S)$.

In contrast, we can easily show that $(0, 1, 2, 3) \in \mathrm{Span}(S)$. If we see a way to write $(0, 1, 2, 3)$ as a linear combination of elements of S, we can just write that down and be done with it. But we can also be systematic. We want to solve the equation $(0, 1, 2, 3) = a(1, 2, 3, 4) + b(1, 1, 1, 1)$, which gives the system

$$\begin{aligned} a + b &= 0 \\ 2a + b &= 1 \\ 3a + b &= 2 \\ 4a + b &= 3 \end{aligned}$$

and the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \\ 4 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus our system has the solution $a = 1, b = -1$, and so $(0, 1, 2, 3) \in \mathrm{Span}(S)$. We can check that indeed $1 \cdot (1, 2, 3, 4) + (-1)(1, 1, 1, 1) = (0, 1, 2, 3)$.

Example 3.11. Let $S = \{\sin^2, \cos^2, \tan^2\}$. Is $1 \in \mathrm{Span}(S)$? Is $\sec^2 \in \mathrm{Span}(S)$?

We know from trigonometry that $\sin^2 + \cos^2 = 1$, so $1 \in \mathrm{Span}(S)$. Then we know that $\sin^2 + \cos^2 + \tan^2 = 1 + \tan^2 = \sec^2 \in \mathrm{Span}(S)$.

Spans are really convenient to work with because the span of any set will always be a subspace.

Proposition 3.12. *If V is a vector space and $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} \subset V$, then $\mathrm{Span}(S)$ is a subspace of V.*

Proof. We know that $S \subset V$, and since any linear combination of vectors in V is itself a vector in V, we know that $\mathrm{Span}(S) \subset V$. So we just need to check the three subspace conditions.

1. We know that $0\mathbf{v} = \mathbf{0}$ for any $\mathbf{v} \in V$. So we have

$$0\mathbf{u}_1 + 0\mathbf{u}_2 + \cdots + 0\mathbf{u}_n = \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}.$$

Thus $\mathbf{0} \in \mathrm{Span}(S)$.

2. Suppose $\mathbf{v}, \mathbf{w} \in \operatorname{Span}(S)$. This implies that we can write

$$\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n \qquad \mathbf{w} = b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n$$

for some $a_i, b_i \in \mathbb{R}$. Thus

$$\begin{aligned}\mathbf{v} + \mathbf{w} &= (a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n) + (b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n) \\ &= (a_1 + b_1)\mathbf{u}_1 + \cdots + (a_n + b_n)\mathbf{u}_n \in \operatorname{Span}(S).\end{aligned}$$

3. Suppose $r \in \mathbb{R}$ and $\mathbf{v} \in \operatorname{Span}(S)$. Then we can write

$$\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n$$

for some $a_i \in \mathbb{R}$. Then

$$r\mathbf{v} = r\left(a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n\right) = (ra_1)\mathbf{u}_1 + \cdots + (ra_n)\mathbf{u}_n \in \operatorname{Span}(S).$$

Thus we see that $\operatorname{Span}(S)$ is a subspace of V. □

Corollary 3.13. *If A is a $m \times n$ matrix, then the rowspace of A is a subspace of $\mathbb{R}^n$.*

As a result of proposition 3.12, we see that every set spans *some* vector space. In fact, this gives us another way to think of the span of a set.

Corollary 3.14. *If V is a vector space and $S \subseteq V$, then $\operatorname{Span}(S)$ is the smallest subspace of V containing S.*

Proof. We just showed in proposition 3.12 that $\operatorname{Span}(S)$ is a subspace of V, and of course it contains S. So we just need to show that there's no smaller subspace. In particular, I'll prove that if W is a subspace of V, and $S \subseteq W$, then $\operatorname{Span}(S) \subseteq W$.

So suppose W is a subspace of V and $S \subseteq W$. Let $\mathbf{v} \in \operatorname{Span}(S)$. The $\mathbf{v}$ is a linear combination of vectors in S. But $S \subseteq W$, so $\mathbf{v}$ is a linear combination of vectors in W, and thus an element of W since W is a vector space. Thus any element of $\operatorname{Span}(S)$ is an element of W, so $\operatorname{Span}(S) \subseteq W$. □

This idea of spanning allows us to generate a set of "coordinates" for a vector space.

Definition 3.15. The set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subseteq V$ is a *spanning set* for V if every vector in V can be written as a linear combination of vectors in S. That is, S is a spanning set for V if $\operatorname{Span}(S) = V$.

Example 3.16. Which of the following are spanning sets for $\mathbb{R}^3$?

1. $\{(1,0,0),(0,1,0),(0,0,1),(2,1,1)\}$
2. $\{(1,0,0),(0,1,0),(2,1,1)\}$
3. $\{(1,1,1),(1,1,0),(0,1,1)\}$
4. $\{(1,1,1),(1,1,0),(0,0,1)\}$
5. $\{(1,2,3),(3,2,1)\}$

We can check this by seeing which vectors we can make as linear combinations of the vectors in each set. Thus for each set, we want to see if we can find coefficients to make (a,b,c) a linear combination of the given vectors.

1.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_4 \\ \alpha_2 + \alpha_4 \\ \alpha_3 + \alpha_4 \end{bmatrix}$$

thus we need to solve the system of equations

$$a = \alpha_1 + 2\alpha_4 \qquad b = \alpha_2 + \alpha_4 \qquad c = \alpha_3 + \alpha_4$$

We can encode this system in an augmented matrix, and get

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & a \\ 0 & 2 & 0 & 1 & b \\ 0 & 0 & 1 & 1 & c \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & a \\ 0 & 1 & 0 & 1/2 & b/2 \\ 0 & 0 & 1 & 1 & c \end{array}\right].$$

This has a free variable, so our solution set is

$$\{(a - 2\alpha_4, b/2 - \alpha_4/2, c - \alpha_4, \alpha_4 : \alpha_4 \in \mathbb{R}\}.$$

Thus our system has a solution for any $a, b, c \in \mathbb{R}$, so this is a spanning set.

You might notice a couple important things from this. First the *columns* of the matrix are the vectors in our set S. This is obvious when you think about it, and makes it easy to write the matrix down. But it also has deep ties to an important result we will cover in section 4.3.

Second, we don't care what the solution is, just whether one exists. If we can get the matrix into row-echelon form and have no self-contradictory rows, then there is *a* solution, and our set spans. So we can basically ignore the right-hand column, since we don't care what the solution actually is.

2. Here we need to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_3 \\ \alpha_2 + \alpha_3 \\ \alpha_3 \end{bmatrix}$$

thus we need to solve the system of equations

$$a = \alpha_1 + 2\alpha_3 \qquad b = \alpha_2 + \alpha_3 \qquad c = \alpha_3.$$

This gives us the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-2c \\ 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & c \end{array}\right]$$

and thus we have the solution $\alpha_1 = a - 2c$, $\alpha_2 = b - c$, and $\alpha_3 = c$. A solution exists, so our set is a spanning set.

Again we notice that the only important thing is that none of our rows consist entirely of zeroes.

3. We can use the same approach. We get the (unaugmented) matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Regardless of the right-hand column, this will have a solution, so once again we have a spanning set.

4. This is very similar to the last problem. We try to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_3 \end{bmatrix}$$

which gives us the three equations

$$a = \alpha_1 + \alpha_2 \qquad b = \alpha_1 + \alpha_2 \qquad c = \alpha_1 + \alpha_3.$$

We immediately see that $a = \alpha_1 + \alpha_2 = b$, so we can't get any vectors with $a \neq b$, so this is not a spanning set.

If we set this up as a matrix, we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix has a row that is all zeroes, so the associated system doesn't have a solution for every (a, b, c).

So what is the span of this set? We saw that $a = b$ for any coefficients we choose, but we can see that we can choose a to be anything we like, and c to be anything we like (e.g. set $\alpha_1 = 0, \alpha_2 = a, \alpha_3 = c$). Thus

$$\operatorname{Span}(S) = \{(a, b, c) : a = b\}$$

which is a plane.

If we wanted to be systematic here, we could write our vectors as the *rows* of a matrix, and row-reduce it. We can look at

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the span of these vectors is $\operatorname{Span}(\{(1, 1, 0), (0, 0, 1)\}) = \{(a, a, c) : a, c \in \mathbb{R}\}$.

5. We have

$$\left[\begin{array}{cc|c} 3 & 1 & a \\ 2 & 2 & b \\ 1 & 3 & c \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & a-b \\ 2 & 2 & b \\ 1 & 3 & c \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & a-b \\ 0 & 4 & 3b-2a \\ 0 & 4 & c-a+b \end{array}\right]$$
$$\to \left[\begin{array}{cc|c} 1 & -1 & a-b \\ 0 & 1 & 3b/4 - a/2 \\ 0 & 1 & c/4 - a/4 + b/4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & a/2 - b/4 \\ 0 & 1 & 3b/4 - a/2 \\ 0 & 0 & c/4 + a/4 - b/2 \end{array}\right].$$

The third row of the unaugmented matrix is all zeroes, so we can't solve this for all possible coefficients. In particular, when we look at the augmented matrix, we see that the third row tells us $0 = c/4 + a/4 - b/2$; thus the equation has solutions if and only if $b/2 = a/4 + c/4$, or $b = (a+c)/2$.

Note that the failure of this set to span is not surprising, since $\mathbb{R}^3$ is "three-dimensional" and we only started with two possible directions to go.

Remark 3.17. Notice that we sometimes want to use our vectors as the rows of a matrix, and sometimes we want to use them as the columns of a matrix. This is a very important duality, and we'll return to it in more detail in section 4.3.

Example 3.18. Which of the following are spanning sets for $\mathcal{P}_3(x)$?

1. $\{1, x, x^2, x^3\}$

   We need to see if we can write an arbitrary polynomial as a linear combination of these elements. So we write

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \alpha_0 \cdot 1 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \alpha_3 \cdot x^3$$

   which gives us the linear equations

$$a_0 = \alpha_0 \qquad a_1 = \alpha_1 \qquad a_2 = \alpha_2 \qquad a_3 = \alpha_3$$

   which...come presolved. So there is a solution to this system, and $\operatorname{Span}(\{1, x, x^2, x^3\}) = \mathcal{P}_3(x)$.

2. $\{1 + x, x^2 + x^3, x + x^2, 1 + x^3\}$

   As before, we try to write a generic polynomial as a linear combination here. We write

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 &= \alpha_0(1+x) + \alpha_1(x^2 + x^3) + \alpha_2(x + x^2) + \alpha_3(1 + x^3) \\ &= (\alpha_0 + \alpha_3) + (\alpha_0 + \alpha_2)x + (\alpha_1 + \alpha_2)x^2 + (\alpha_1 + \alpha_3)x^3 \end{aligned}$$

   which gives us the system of equations

$$a_0 = \alpha_0 + \alpha_3 \qquad a_1 = \alpha_0 + \alpha_2 \qquad a_2 = \alpha_1 + \alpha_2 \qquad a_3 = \alpha_1 + \alpha_3.$$

We can turn this system of equations into a matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a_0 \\ 1 & 0 & 1 & 0 & a_1 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 1 & 0 & 1 & a_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a_0 \\ 0 & 0 & 1 & -1 & a_1 - a_0 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 0 & -1 & 1 & a_3 - a_2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a_0 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 0 & 1 & -1 & a_1 - a_0 \\ 0 & 0 & 0 & 0 & a_3 - a_2 + a_1 - a_0 \end{array}\right]$$

We get a row of all zeroes, meaning that the set doesn't span. In particular, we get the constraint that $a_3 - a_2 + a_2 - a_0 = 0$; thus the span is the set of all polynomials where the sum of the even-degree coefficients is the same as the sum of the odd-degree coefficients. And we can go back and check that this is a property that all of our original vectors have, and that is stable under addition and scalar multiplication.

Obviously answering this question effectively requires a thorough study of solving systems of equations like this; we will return to this question in great detail in the future.

We finish with a few facts about spans and spanning sets:

Proposition 3.19. *Let V be a vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Then*

1. $\mathbf{0} \in \operatorname{Span}(S)$.
2. *If $S \subseteq T$ then* $\operatorname{Span}(S) \subseteq \operatorname{Span}(T)$.
3. *If $\mathbf{u} \in \operatorname{Span}(S)$, then we write $S \cup \{\mathbf{u}\} = \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}\}$ for the set containing everything in S, and also the element $\mathbf{u}$. Then*
$$\operatorname{Span}(S) = \operatorname{Span}(S \cup \{\mathbf{u}\}).$$
4. *If W is a subspace of V then* $\operatorname{Span}(W) = W$.

Proof. 1. $0 = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n \in \operatorname{Span}(S)$.

2. Set $T = \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1, \dots, \mathbf{u}_m\}$. Suppose $\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \in \operatorname{Span}(S)$. Then
$$\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + 0\mathbf{u}_1 + \dots 0\mathbf{u}_m \in \operatorname{Span}(T).$$

3. To prove two sets are equal, it's generally easiest to prove each one is a subset of the other—that is, for each set, we take an arbitrary element of that set and prove it is also an element of the other set.

   We know that $S \subseteq S \cup \{\mathbf{u}\}$, so by part (2) we know that $\operatorname{Span}(S) \subseteq \operatorname{Span}(S \cup \{\mathbf{u}\})$.

   So suppose $\mathbf{w} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n + c\mathbf{u} \in \operatorname{Span}(S)$. We know $\mathbf{u} \in \operatorname{Span}(S)$ so we can write $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. Then

$$\begin{aligned}\mathbf{w} &= b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n + c\mathbf{u} \\ &= b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n + c\,(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) \\ &= (b_1 + ca_1)\mathbf{v}_1 + \cdots + (b_n + ca_n)\mathbf{u}_n \in \operatorname{Span}(S).\end{aligned}$$

   Thus $\operatorname{Span}(S) \subseteq \operatorname{Span}(S \cup \{\mathbf{u}\})$ and $\operatorname{Span}(S \cup \{\mathbf{u}\}) \subseteq \operatorname{Span}(S)$, so we know that $\operatorname{Span}(S) = \operatorname{Span}(S \cup \{\mathbf{u}\})$.

4. We know that $W \subseteq \operatorname{Span}(W)$, so we just need to show that $\operatorname{Span}(W) \subseteq W$. Let $\mathbf{w} = a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n \in \operatorname{Span}(W)$, where $\mathbf{w}_1, \dots, \mathbf{w}_n \in W$. Then since W is a vector space, it is closed under linear combinations, so any linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_n$ is in W. Thus in particular $\mathbf{w} \in W$.

□

Corollary 3.20. *If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a spanning set for a vector space V, and $T = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a set with $S \subseteq \operatorname{Span}(T)$, then T is also a spanning set for V.*

Proof. We know that $S \subseteq \operatorname{Span}(T)$, and this means that $\operatorname{Span}(S) \subseteq \operatorname{Span}(\operatorname{Span}(T))$. But $\operatorname{Span}(\operatorname{Span}(T)) = \operatorname{Span}(T)$, so we have that $V = \operatorname{Span}(S) \subseteq \operatorname{Span}(T)$. Thus T spans V.

□

3.2 Linear Independence

This idea of spanning sets answers half of our original question. If we have a spanning set for V, we can write our vectors as sums of elements of the spanning set. But recall we also want this representation to be *unique*—we want to know that if we give two different sets of "coordinates" that they actually represent distinct vectors.

In the previous section, we wanted to study the span of a set of vectors—which tells you how many places you can get with them. Now we want to measure the redundancy: Do we have more vectors in our spanning set than we need?

Definition 3.21. We say a set of vectors $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq V$ is *linearly independent* if the only scalars solving the equation

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$$

are $a_1 = \cdots = a_n = 0$.

If a set of vectors is not linearly independent, we call it *linearly dependent* and there is a *linear dependence* relationship among the vectors.

Example 3.22.

1. The set $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly independent: suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Then we have the system of equations $a = 0, b = 0, c = 0$ and thus all the scalars are zero.

2. The set $S = \{(1,0,0),(0,1,0)\}$ is linearly independent. Suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}.$$

Then we have the system of equations $a = 0, b = 0$ and thus all the scalars are zero.

3. The set S $= \{(1,0,0),(0,1,0),(1,1,0)\}$ is not linearly independent, since

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$

4. Any set containing the zero vector is linearly dependent, since $1 \cdot \mathbf{0} = \mathbf{0}$ but $1 \neq 0$.

As before, we see that each problem is really asking for the solution to a system of equations. But unlike in section 3.1 we don't need to find a solution for any possible constants. Instead, we just want to see if there's more than one solution to the equation $A\mathbf{x} = \mathbf{0}$.

Example 3.23. Let $S = \{(3,5,1),(3,5,3),(1,1,1)\}$. Then we want to know if there exist a, b, c such that

$$a \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The associated matrix is $A = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ and we want to solve the homogeneous system, so we just have to reduce the matrix: we get

$$\begin{bmatrix} 3 & 3 & 1 \\ 5 & 5 & 1 \\ 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & -6 & -2 \\ 0 & -10 & -4 \\ 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And thus we have the unique solution $a = 0, b = 0, c = 0$. Thus S is linearly independent.

There are a few ways we can think about linear independence. One is that a linearly independent set is one where the zero vector can be expressed uniquely—$\mathbf{0}$ is in the span of any set, but it is only in the span of a linearly independent set in one way. In fact, this is enough to make *every* vector expressed uniquely.

Proposition 3.24. *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a subset of a vector space V. Then S is linearly independent if and only if every vector in* Span(S) *can be expressed uniquely as a linear combination of vectors in S.*

Proof. Suppose S is linearly dependent. Then by definition of linear independence there are $a_i \in \mathbb{R}$ such that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$$

and thus the expression of $\mathbf{0}$ as a linear combination of vectors in S is not unique.

Now suppose not every vector in Span(S) can be expressed uniquely as a linear combination of vectors in S. By definition of span, every vector in Span(S) can be represented as a linear combination of vectors in S, so it must be the case that some vector is not represented uniquely, and thus can be written as a linear combination of elements of S in two different ways.

Suppose $\mathbf{u}$ is such an element. Then we have

$$\begin{aligned} a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n &= \mathbf{u} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n \\ (a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n &= \mathbf{u} - \mathbf{u} = \mathbf{0}. \end{aligned}$$

Thus we can write $\mathbf{0}$ as a nontrivial linear combination of elements of S, so S is linearly dependent. $\square$

Another way to think of this is that in a linearly dependent set, we can express one vector as a linear combination of the others, and thus at least one vector in the set is redundant.

This gives us a geometric interpretation as well. Generally, any one vector defines a line containing it and the origin. Two vectors in general define a plane, three vectors a threespace, and so on. A set is linearly independent if the linear space it defines is as big as you would expect. A set is linearly dependent if the set is smaller—if, say, you have points but they're all on the same line through the origin, so you don't actually get a whole plane.

Lemma 3.25. *A set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if some element can be written as a linear combination of the others.*

Proof. Without loss of generality, assume $\mathbf{v}_1$ can be written as a linear combination of $\mathbf{v}_2, \dots, \mathbf{v}_n$. (That is, we're assuming one of the vectors can be written as a linear combination of the others, and since order doesn't matter we can assume that it's $\mathbf{v}_1$ to keep the notation simple). Then we have

$$\begin{aligned} \mathbf{v}_1 &= a_2\mathbf{v}_2 + \dots a_n\mathbf{v}_n \\ \mathbf{v}_1 - \mathbf{v}_1 &= a_2\mathbf{v}_2 + \dots a_n\mathbf{v}_n - \mathbf{v}_1 \\ \mathbf{0} &= (-1)\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n. \end{aligned}$$

Then we have written $\mathbf{0}$ as a nontrivial linear combination of elements of S, and thus S is linearly dependent.

Conversely, suppose S is linearly dependent. Then there are a_i not all zero such that

$$\mathbf{0} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n.$$

We know not all the a_i are zero, so assume without loss of generality that $a_1 \neq 0$. Then we have

$$\begin{aligned} -a_1\mathbf{v}_1 &= a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n \\ \mathbf{v}_1 &= \frac{-a_2}{a_1}\mathbf{v}_2 + \dots + \frac{-a_n}{a_1}\mathbf{v}_n \end{aligned}$$

and thus we can write $\mathbf{v}_1$ as a linear combination of the other vectors in S. $\square$

Corollary 3.26. *$S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if there is some $\mathbf{v}_i$ such that* $\operatorname{Span}(S) = \operatorname{Span}(\{\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n\}$.

In practice this is how we prefer to test for linear independence: we try to write one vector as a linear combination of the others. Sometimes this is easy and we're done. Other times this is difficult, or we become convinced it's not possible, and then we have to go back to solving linear equations.

Example 3.27. 1. Let $S = \{(1,1,1),(1,1,0),(0,0,1)\}$. We see that $(1,1,1) = (1,1,0) + (0,0,1)$ so this set is linearly dependent.

2. Let $S = \{(1,1,1),(1,1,0),(0,1,1)\}$. It might look like this is similar, and we could write $(1,1,1)$ somehow as a combination of the other two. But we see that's not actually possible. In fact we write

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ a+b+c \\ b+c \end{bmatrix}$$

and this gives us the system

$$0 = a+b \qquad 0 = a+b+c \qquad 0 = b+c$$

with associated matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus we get the unique solution $a = b = c = 0$ and so the set is linearly independent.

3. Let $S = \{(1,1,1),(1,1,0),(2,3,1),(0,1,1)\}$. To show linear dependence, we might want to show that one vector is a sum of the others. In fact we cannot write $(1,1,1)$ as a linear combination of the other vectors:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+2b \\ a+3b+c \\ b+c \end{bmatrix}$$

gives the system

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

and the last row is a contradiction $0 = 1$. Thus there is no solution to this system; we cannot write $(1, 1, 1)$ as a linear combination of the other vectors.

But this doesn't mean that the vectors are linearly independent. Corollary 3.26 says that the vectors are independent if and only if *some* vector is a linear combination of the others. There is in fact a vector that is a sum of some of the others: we see that

$$2\begin{bmatrix}1\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\3\\1\end{bmatrix}$$

and thus set S is linearly dependent.

If we didn't see this directly, we could set up the matrix associated to all four vectors:

$$\begin{bmatrix}1&1&2&0\\1&1&3&1\\1&0&1&1\end{bmatrix} \to \begin{bmatrix}1&1&2&0\\0&0&1&1\\0&-1&-1&1\end{bmatrix} \to \begin{bmatrix}1&0&1&1\\0&1&1&-1\\0&0&1&1\end{bmatrix} \to \begin{bmatrix}1&0&0&0\\0&1&0&-2\\0&0&1&1\end{bmatrix}.$$

We have the fourth column as a free variable, so there is more than one solution. Thus the set is not linearly independent. In particular, if we have a linear combination of these vectors that sums to $\mathbf{0}$, it satisfies $a = 0, b = 2d, c = -d$.

Again we see we're looking at a matrix whose columns are the vectors in the set in question. We want to see if any non-trivial linear combination of our vectors is equal to the zero vector, and that leads to solving a homogeneous system of equations. Thus we get the following result:

Proposition 3.28. *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset \mathbb{R}^m$. Let A be the matrix whose columns are the vectors $\mathbf{v}_i$. Then S is linearly independent if and only if $N(A) = \{\mathbf{0}\}$.*

Proof. By definition, S is linearly independent if and only if the only solution to $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$ is $(0, 0, \dots, 0)$. But this is a homogeneous system of linear equations whose matrix is A, so $N(A)$ is the set of solutions to this system of equations. Thus S is linearly independent if and only if the only element of $N(A)$ is the zero vector. □

We again conclude with a fact about linear independence.

Proposition 3.29. *If $S \subseteq T$ and T is linearly independent, then S is also linearly independent.*

Proof. Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1, \dots, \mathbf{u}_m\} = T$, and T is linearly independent. Now suppose their are scalars a_i such that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}.$$

Then we have

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + 0\mathbf{u}_1 + \cdots + 0\mathbf{u}_m = 0$$

and since T is linearly independent, we have $a_i = 0$ for every a_i. Thus we see that S is linearly independent.

□

3.3 Vector Space Bases

Having now discussed the two properties we want a coordinate system to have, we can define exactly what we mean by a coordinate system.

Definition 3.30. If V is a vector space and S is a spanning set for V that is also linearly independent, we say that S is a *basis* for V.

Example 3.31. The set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathbb{R}^3$, as we have seen before. We call this set the *standard basis* for $\mathbb{R}^3$, and we write the three elements $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

We can generalize this to $\mathbb{R}^n$. We define the *standard basis vectors* for $\mathbb{R}^n$ by

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \dots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and the set of standard basis vectors is the *standard basis*. You can check that the standard basis is in fact a basis.

Example 3.32. Every (non-trivial) vector space has more than one basis. The set $S = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis for $\mathbb{R}^3$:

First we show that it is a spanning set. Let $(a,b,c) \in \mathbb{R}^3$. Then we want to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which gives the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-b \\ 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & c \end{array}\right]$$

which tells us that $\alpha_3 = c, \alpha_2 = b-c, \alpha_1 = a-b$. Thus there is a solution for any $(a,b,c) \in \mathbb{R}^3$, and the set spans.

We also need to prove linear indepencence. So suppose

$$\mathbf{0} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This gives us a system of linear equations corresponding to the homogeneous system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the only solution here is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Thus S is linear independent, and since it also spans, it is a basis.

Example 3.33. The set $S = \{(1,0,0),(0,1,0)\}$ is not a basis for $\mathbb{R}^3$. It is linearly independent (since it is a subset of the standard basis, which is linear independent), but it is not a spanning set, since $(0,0,1)$ is not in the span of S.

Example 3.34. The set $S = \{(2,3),(3,4),(4,4)\}$ is a spanning set for $\mathbb{R}^2$ but not a basis. To see that it's a spanning set we solve

$$\begin{bmatrix} a \\ b \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \\ 3\alpha_1 + 4\alpha_2 + 4\alpha_3 \end{bmatrix}$$

giving the system of equations

$$a = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \qquad b = 3\alpha_1 + 4\alpha_2 + 4\alpha_3$$

and the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & a \\ 3 & 4 & 4 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & b-a \\ 2 & 3 & 4 & a \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & b-a \\ 0 & 1 & 4 & 3a-2b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -4 & 3b-4a \\ 0 & 1 & 4 & 3a-2b \end{array}\right].$$

Thus for any $(a, b) \in \mathbb{R}^2$, at least one solution exists; in fact we can pick α_3 to be any real number and we get a corresponding solution $(3b - 4a + 4\alpha_3, 3a - 2b - 4\alpha_3, \alpha_3)$. Thus the set spans.

But S is not linearly independent. We can see this in a few ways. Most easily we can observe that $(2, 3) + (1/4)(4, 4) = (3, 4)$. If we can't see that on our own, we can do a couple things. We can find the nullspace:

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \end{bmatrix}$$

and we see the nullspace $\{(4\alpha, -4\alpha, \alpha)\}$ is non-trivial, so the set is not linearly independent.

But if these row operations seem familiar, that's because we did exactly the same thing to check spanning! So we can look at our spanning equations and try to find all the solutions when we take $a = b = 0$. We see that there's more than one solution there, so the vectors aren't linearly independent.

Example 3.35. The set $S = \{1, x, x^2, x^3\}$ is a basis for $\mathcal{P}_3(x)$. So is the set $T = \{1 + x + x^2 + x^3, 1 + x + x^2, 1 + x, 1\}$.

Determining whether a set is a basis is sometimes annoying, but doesn't involve anything we haven't already done: a basis is just a set that both spans and is linearly independent, and we can check both properties individually. But we'd like to make things a little simpler.

Further, we want to talk about how "big" a space is, and this should plausibly be determined by how many elements there are in the basis. But since every space has more than one basis, talking about the size of "the" basis is potentially problematic. Fortunately, this is not an actual problem, as we shall see.

Lemma 3.36. *If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans a vector space V, and $T = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a collection of vectors in V with $m > n$, then T is linearly dependent.*

Proof. There are two possible ways to prove this. One involves simply writing out a bunch of linear equations and solving them; this works, but is more tedious than informative. We'll use a more formal and abstract approach to proving this instead, which, hopefully, will actually explain some of *why* this is true.

We will start with the set S, and one by one we will trade out vectors in S for vectors in T, and show that we always still have a spanning set. We will suppose T is linearly independent, and show that $m \leq n$.

Since S is a spanning set, we know that $\mathbf{u}_1 \in \operatorname{Span}(S)$, and thus $\{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1\}$ is linearly dependent by lemma 3.25. Then we can rewrite our linear dependence equation to express $\mathbf{v}_1$ (without loss of generality) as a linear combination of $\{\mathbf{u}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = S_1$, and thus

$$\operatorname{Span}(S) = \operatorname{Span}(\{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1\}) = \operatorname{Span}(S_1).$$

We can repeat this process: at every step we add the next vector from T to get the set $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_k, \dots, \mathbf{v}_n\}$. Since S_{k-1} is a spanning set, this set is linearly dependent; since the $\mathbf{u}_i$ are linearly independent by hypothesis, we can remove one of the $\mathbf{v}_i$, and without loss of generality we can remove $\mathbf{v}_k$, to obtain the set $S_k = \{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$.

If $m > n$, we can continue until we have replaced every $\mathbf{v}_i$. Then we have $S_n = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a spanning set, and thus $\mathbf{u}_{n+1} \in \operatorname{Span}(S_n)$ and so T is linearly dependent, which contradicts our assumption.

Thus if T is linearly independent, we must have $m \leq n$. Conversely, if $m > n$ then T is linearly dependent, as we stated. □

Corollary 3.37. *$S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $T = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ are two bases for a space V, then they are the same size, i.e. $m = n$.*

Proof. S is a spanning set and T is linearly independent, so we can't have $m > n$ by lemma 3.36. But T is a spanning set and S is linearly independent, so we can't have $n > m$ by lemma 3.36. Thus $n = m$. □

Definition 3.38. Let V be a vector space. If V has a basis consisting of n vectors, we say that V has *dimension* n and write $\dim V = n$. The trivial vector space $\{\mathbf{0}\}$ has dimension 0.

We say that V is *finite–dimensional* if there is a finite set of vectors that spans V. (Thus if V is n-dimensional it is finite-dimensional). Otherwise, we say that V is *infinite–dimensional.*

In this course we will primarily discuss finite dimensional vector spaces.

Example 3.39. The set of standard basis vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$, so $\mathbb{R}^n$ is n-dimensional.

The set $\{1, x, \dots, x^n\}$ is a basis for $\mathcal{P}_n(x)$. This set has $n+1$ vectors, so $\dim \mathcal{P}_n(x) = n+1$.

$\mathcal{P}(x)$ does not have a finite basis. We can see this since the set $S = \{1, x, \dots, x^n\}$ is linearly independent for any n; but every spanning set is at least as big as any linearly independent set, so we can never have a finite spanning set. However, if we allow infinite bases, then $\{1, x, \dots, x^n, \dots\}$ is a basis for $\mathcal{P}(x)$.

Remark 3.40. $\mathcal{C}([a,b],\mathbb{R})$ is infinite–dimensional, but if we allow infinite sums and make convergence arguments it is possible to think of the set $\{1, x, \dots, x^n, \dots\}$ as a sort of ("separable") basis. But this requires analysis and is outside the scope of this course. We can also build a (separable) basis out of the functions $\sin(nx)$ and $\cos(nx)$ for $n \in \mathbb{N}$; this is the foundation of Fourier analysis and Fourier series.

The set $\mathcal{F}(\mathbb{R},\mathbb{R})$ is absurdly huge, and does not have a countable basis. If you believe the axiom of choice it has a basis, as all sets do, but you can't possibly write it down. You can think of it has having "coordinates" given by functions like

$$f_r(x) = \begin{cases} 1 & x = r \\ 0 & x \neq r \end{cases}$$

but this isn't a basis because it would require uncountable sums, which you can't really define.

We'd like to make it easier to check if a set is a basis, and easier to find bases for spaces. We show here that if we start with basically any set, we can turn it into a basis.

Lemma 3.41 (Basis Reduction). *Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a spanning set for V. Then S can be reduced to a basis for V. That is, there is a subset $B \subseteq S$ that is a basis for V.*

Proof. If S is linearly independent, then it is a basis and we're done.

So suppose S is linearly dependent. Then we know at least one vector is redundant, so without loss of generality we can reorder the set so that we can write $\mathbf{v}_n$ as a linear combination of the other vectors in S.

But then $\operatorname{Span}(S) = \operatorname{Span}(\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\})$, and $S_1 = \{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ is a spanning set for V and a proper subset of S. If S_1 is linearly independent, then it is a basis; if not, we can repeat this process until we reach a linearly independent set, which is our basis B. □

Remark 3.42. This proof assumes that S is finite. The result is still (mostly) true if S is infinite, but if the space is finite-dimensional this isn't interesting, and if the space is infinite-dimensional things get very complicated and we don't want to worry about them here.

Example 3.43. Let $S = \{(1,1,0), (1,1,1), (0,0,1), (2,7,0)\}$ be a spanning set for $\mathbb{R}^3$. Find a basis $B \subseteq S$ for $\mathbb{R}^3$.

We'll take as given that this is a spanning set, which is not difficult to check. We see that we can write $(1,1,1) = (1,1,0) + (0,0,1)$, so we can remove $(1,1,1)$ without changing the span, and we have $B = \{(1,1,0), (0,0,1), (2,7,0)\} \subseteq S$ is a basis for $\mathbb{R}^3$.

Example 3.44. Let $S = \{(1,2,3),(1,1,1),(5,-2,1),(-4,3,2)\}$ be a spanning set for $\mathbb{R}^3$. Find a basis $B \subseteq S$ for $\mathbb{R}^3$.

We'll take as given that S is a spanning set. We need to write one vector as a linear combination of the others, which is essentially the same problem as finding a nontrivial linear combination equal to zero. So we set up the equation

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix} + c\begin{bmatrix}5\\-2\\1\end{bmatrix} + d\begin{bmatrix}-4\\3\\2\end{bmatrix}$$

which gives us

$$\begin{bmatrix}1&1&5&-4\\2&1&-2&3\\3&1&1&2\end{bmatrix} \to \begin{bmatrix}1&1&5&-4\\0&-1&-12&11\\0&-2&-14&14\end{bmatrix} \to \begin{bmatrix}1&0&-7&7\\0&1&12&-11\\0&0&10&-8\end{bmatrix} \to \begin{bmatrix}1&0&0&7/5\\0&1&0&-7/5\\0&0&1&-4/5\end{bmatrix}$$

This gives us $a = -7/5d, b = 7/5d, c = 4/5d$, or in other words if we set $d = 1$ we get

$$\begin{bmatrix}-4\\3\\2\end{bmatrix} = \frac{7}{5}\begin{bmatrix}1\\2\\3\end{bmatrix} - \frac{7}{5}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{4}{5}\begin{bmatrix}5\\-2\\1\end{bmatrix}.$$

Thus $(-4,3,2)$ can be written as a linear combination of the other vectors, and so we have $B = \{(1,2,3),(1,1,1),(5,-2,1)\}$ is a basis for $\mathbb{R}^3$. We know this is a basis because it is still a spanning set, and has the correct number of elements.

(We could actually have removed any vector from this set and still gotten a basis; each element can be written as a combination of the others, as you can see by rearranging the last equation. But it's sufficient here to find one basis.)

Example 3.45. Let $S = \{1-x, x^2-x, 1+x+x^2, x^2-1\}$ be a spanning set for $\mathbb{P}_2(x)$. Find a basis $B \subset S$.

We need to remove one vector which depends on the others. We need to find a nontrivial linear combination, so we have the equation

$$a(1-x) + b(x^2-x) + c(1+x+x^2) + d(x^2-1) = 0$$

which gives the homogeneous system

$$\begin{bmatrix}1&0&1&-1\\-1&-1&1&0\\0&1&1&1\end{bmatrix} \to \begin{bmatrix}1&0&1&-1\\0&-1&2&-1\\0&1&1&1\end{bmatrix} \to \begin{bmatrix}1&0&1&-1\\0&1&1&1\\0&0&3&0\end{bmatrix} \to \begin{bmatrix}1&0&0&-1\\0&1&0&1\\0&0&1&0\end{bmatrix}$$

which tells us that $a = d, b = -d, c = 0$.

Thus we have $(1-x) - (x^2 - x) + (x^2 - 1) = 0$ and so $x^2 - x = (1-x) + (x^2 - 1)$, and thus the element $x^2 - x$ is redundant and a linear combination of the other vectors. We can remove it, and get a basis $\{1 - x, 1 + x + x^2, x^2 - 1\}$.

Notice here that we could have removed the first element $1 - x$ or the fourth element $x^2 - 1$, since we can rearrange our equation to write either of those as a linear combination of the others. But we could *not* have removed the element $1 + x + x^2$, since we didn't find we could write it as a combination of the others; it was in fact necessary for this set to span.

Lemma 3.46 (Basis Padding). *Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent in V. Then if V has any finite spanning set $T = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$, we can obtain a basis by padding S. That is, there is a basis B for V with $S \subseteq B$.*

Proof. If $T \subset \text{Span}(S)$, then $\text{Span}(T) \subset \text{Span}(S)$, so S is a spanning set for V and thus a basis, so we're done.

So suppose without loss of generality that $\mathbf{u}_1 \notin \text{Span}(S)$. Then $S_1 = \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1\}$ is linearly independent by lemma 3.25 since we can't write any element as a linear combination of the others.

If S_1 spans V, then it is a basis and we're done. If not, there is some other $\mathbf{u}_i \notin \text{Span}(S_1)$, so we can repeat the process, and after at most m steps this process will terminate (since we run out of elements in T). When we reach a spanning set, this is our basis.

□

Example 3.47. Let $S = \{(1,1,0), (1,-1,0)\}$. Find a basis $B \supseteq S$ for $\mathbb{R}^3$.

We see that S is linearly independent, so we just need to find a vector that isn't in $\text{Span}(S)$. It's clear that $(0,0,1) \notin \text{Span}(S)$, so we see that $B = \{(1,1,0), (1,-1,0), (0,0,1)\}$ satisfies our requirements.

But there are many choices we could make. It's also the case that $(1,1,1) \notin \text{Span}(S)$, so we see that $B_1 = \{(1,1,0), (1,-1,0), (1,1,1)\}$ also satisfies our requirements.

Example 3.48. Let $T = \{(5,2,-3), (1,-4,7)\}$. Find a basis $B \supseteq T$.

We just need to find a vector that isn't in $\text{Span}(T)$. We can make a guess here and prove it by hand; so for instance it looks like $(1,0,0)$ is not in the span. Indeed, we see that if

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} + b \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$$

then we have the system

$$\left[\begin{array}{cc|c} 5 & 1 & 1 \\ 2 & -4 & 0 \\ -3 & 7 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 9 & 1 \\ 0 & -22 & -2 \\ 0 & 34 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2/11 \\ 0 & 1 & 1/11 \\ 0 & 0 & -1/11 \end{array}\right]$$

and thus we have a contradiction.

Thus $(1, 0, 0)$ is not in the span of T, and we have that $B = \{(5, 2, -3), (1, -4, 7), (1, 0, 0)\} \supseteq T$ is a basis for $\mathbb{R}^3$.

Example 3.49. Let $S = \{1 + x, x^2 - 3\} \subset \mathcal{P}_2(x)$. Can we find a basis B for $\mathcal{P}_2(x)$ that contains T?

We need to find a vector (or quadratic polynomial) that isn't in S. There are lots of choices here, but it looks to me like 1 is not in the span of S. Then we check: suppose $a(1 + x) + b(x^2 - 3) = 1$. Then we have

$$(a - 3b) + ax + bx^2 = 1$$

which gives the system

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

which has no solution. Thus indeed $1 \notin \text{Span}(S)$, so $\{1, 1 + x, x^2 - 3\}$ is a basis for $\mathcal{P}_3(x)$.