7 Eigenvectors and Eigenvalues

In this section we will study a special type of basis, called an eigenbasis. For (almost) any given operator, we get a specific basis which will make most our computations easier.

7.1 Eigenvectors

Definition 7.1. Let $L : V \to V$ be a linear transformation, and let λ be a scalar. If there is a vector $\mathbf{v} \in V$ such that $L\mathbf{v} = \lambda\mathbf{v}$, then we say that λ is an *eigenvalue* of L, and $\mathbf{v}$ is an *eigenvector* with eigenvalue λ.

Geometrically, an eigenvector corresponds to a direction in which our linear operator purely stretches or shrinks vectors, without rotating or reflecting them at all. It can often be an axis of rotation.

Example 7.2. Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. We can check that if $\mathbf{x} = (2, 1)$, then

$$A\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so $\mathbf{x}$ is an eigenvector with eigenvalue 3. Similarly, we can check that if $\mathbf{y} = (1, 1)$, then

$$A\mathbf{y} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus $\mathbf{y}$ is an eigenvector with eigenvalue 2.

Example 7.3. Let $R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation map. We can see geometrically that this has no non-trivial eigenvectors, since it changes the direction of any vector. Algebraically, if (x, y) is an eigenvector, then we would have

$$R_{\pi/2}(x, y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

and thus weh ave $\lambda y = x, \lambda x = -y$, and the only solution here is $x = y = 0$.

In contrast, if we take the rotation map $R : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates around the z-axis, the vector $(0, 0, 1)$ will be an eigenvector with eigenvalue 1.

Example 7.4. Let $V = \mathcal{D}(\mathbb{R}, \mathbb{R})$ be the space of differentiable real functions, and let $\frac{d}{dx} : V \to V$ be the derivative map. If $f(x) = e^{rx}$, then $\frac{d}{dx} f(x) = re^{rx} = rf(x)$, so f is an eigenvector with eigenvalue r.

Proposition 7.5. *Let V be a vector space and $L : V \to V$ a linear transformation. $\mathbf{v}$ is an eigenvector with eigenvalue λ if and only if $\mathbf{v} \in \ker(L - \lambda I)$.*

Proof. $\mathbf{v}$ is an eigenvector with eigenvalue λ if and only if $L\mathbf{v} = \lambda \mathbf{v} = \lambda I \mathbf{v}$, if and only if $\mathbf{0} = L\mathbf{v} - \lambda I \mathbf{v} = (L - \lambda I)\mathbf{v}$, if and only if $\mathbf{v} \in \ker(L - \lambda I)$. □

Corollary 7.6. *The set of eigenvectors with eigenvalue λ is a subspace of V, called the* eigenspace *corresponding to λ. We denote this space E_λ.*

Corollary 7.7. *A transformation L is invertible if and only if 0 is not an eigenvalue of L.*

Proposition 7.8. *Let $L : V \to V$ be a linear transformation. If $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a set of eigenvectors each with a distinct eigenvalue, then E is linearly independent.*

Proof. Let λ_i be the eigenvalue corresponding to $\mathbf{e}_i$. If E is linearly dependent, then we can write

$$a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n = \mathbf{0}$$

for some nonzero a_i. Let k be the smallest positive integer such that $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ is linearly dependent; then we must have $a_k \neq 0$, and we can compute

$$\begin{aligned}
\mathbf{e}_k &= \frac{-a_1}{a_k}\mathbf{e}_1 + \cdots + \frac{-a_{k-1}}{a_k}\mathbf{e}_{k-1} \\
L(\mathbf{e}_k) &= L\left(\frac{-a_1}{a_k}\mathbf{e}_1 + \cdots + \frac{-a_{k-1}}{a_k}\mathbf{e}_{k-1}\right) = \frac{-a_1}{a_k}L(\mathbf{e}_1) + \cdots + \frac{-a_{k-1}}{a_k}L(\mathbf{e}_{k-1}) \\
\lambda_k\mathbf{e}_k &= \frac{-a_1}{a_k}\lambda_1\mathbf{e}_1 + \cdots + \frac{-a_{k-1}}{a_k}\lambda_{k-1}\mathbf{e}_{k-1}.
\end{aligned}$$

We can multiply the first equation by λ_1 and subtract from the last equation; this gives us

$$\mathbf{0} = \frac{-a_1}{a_k}(\lambda_1 - \lambda_k)\mathbf{e}_1 + \cdots + \frac{-a_{k-1}}{a_k}(\lambda_{k-1} - \lambda_k)\mathbf{e}_{k-1}.$$

But we know by hypothesis that the set $\{\mathbf{e}_1, \dots, \mathbf{e}_{k-1}\}$ is linearly independent, so all these coefficients must be zero. Since the a_i are not all zero, we must have at least some $\lambda_i - \lambda_k = 0$. □

It's straightforward enough to *check* that a vector is an eigenvector if we already have a candidate; but how do we find them? Sometimes this is easy

Example 7.9. Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. What are the eigenvalues and eigenspaces of A?

We see that

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}.$$

Thus the eigenvalues are 3 and 2; the corresponding eigenspaces are spanned by $(1, 0)$ and $(0, 1)$, respectively.

When things aren't this easy, there is still a fairly straightforward approach we can take:

Example 7.10. Let $B = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix}$. Find the eigenvalues and eigenvectors of B.

If $\mathbf{x} = (x, y)$ is an eigenvector with eigenvalue λ, then we have

$$B\mathbf{x} = \begin{bmatrix} 7x + 2y \\ 3x + 8y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

so we have the system of equations $7x + 2y = \lambda x, 3x + 8y = \lambda y$. Equivalently, we have $(7 - \lambda)x + 2y = 0$ and $(3x + (8 - \lambda)y = 0$. We row-reduce

$$\begin{bmatrix} 7-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} \to \begin{bmatrix} 3 & 8-\lambda \\ 0 & 2 + (8-\lambda)(\lambda - 7)/3 \end{bmatrix}$$

$$\to \begin{bmatrix} 3 & 8-\lambda \\ 0 & 6 + (-56 + 15\lambda - \lambda^2) \end{bmatrix} = \begin{bmatrix} 3 & 8-\lambda \\ 0 & -\lambda^2 + 15\lambda - 50 \end{bmatrix}.$$

We first see that this is solveable if and only if $0 = \lambda^2 - 15\lambda + 50 = (\lambda - 5)(\lambda - 10)$, and thus if $\lambda = 5$ or $\lambda = 10$. Thus these are the two eigenvalues for B.

If $\lambda = 5$ then we have $3x + 3y = 0$ so $y = -x$. Any vector $(\alpha, -\alpha)$ will be an eigenvector with eigenvalue 5, so the eigenspace for 5 is the span of $\{(1, -1)\}$. And indeed, we compute

$$B(1, -1) = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

If $\lambda = 10$ then we have $3x - 2y = 0$ and $y = 3/2x$. Thus any vector $(2\alpha, 3\alpha)$ will be an eigenvector with eigenvalue 10, and the corresponding eigenspace is spanned by $\{(2, 3)\}$. We check:

$$B(2, 3) = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix} = 10 \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

As the previous example shows, it is completely possible to find the eigenvectors and eigenvalues with the tools we have already, but it's pretty fiddly even for a small example. We'd like to streamline the process, and this leads us to define the determinant.

7.2 Determinants

Definition 7.11. Let $A \in M_{n\times n}$. If A has n distinct eigenvalues, we say that the *determinant* of A, written $\det A$, is the product of the eigenvalues.

More generally, the determinant of A is the product of the eigenvalues "up to multiplicity". Thus if the eigenspace of $\lambda = 2$ is three-dimensional, we will multiply in λ three times.

Definition 7.12 (Formal definition we won't really use)**.**

$$\det A = \prod_\lambda \lambda^{e_\lambda} \qquad \text{where} \quad e_\lambda = \dim\ker(A - \lambda I)^n.$$

The determinant is (roughly) the product of the eigenvalues, so it can tell something about what the eigenvalues are. But this doesn't help if we don't have a way of finding the determinant without already knowing the eigenvalues. Fortunately, there is a simple way to compute it.

Example 7.13. The determinant of $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ is $3 \cdot 2 = 6$.

The determinant of $B = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix}$ is $5 \cdot 10 = 50$.

Geometrically, the determinant represents the volume of the n-dimensional solid that our matrix sends the n-dimensional unit cube to; thus it tells us how much our matrix stretches its inputs.

7.2.1 The Laplace Formula

We first need to develop some notation.

Definition 7.14. Let $A = (a_{ij})$ be a $n \times n$ matrix. We define the *i,jth minor matrix* of A to be the $(n-1)\times(n-1)$ matrix M_{ij} obtained by deleting the row and column containing a_{ij}—that is, deleting the ith row and jth column.

We define the *i,jth minor* of A to be $\det M_{ij}$. We define the *i,jth cofactor* to be $A_{ij} = (-1)^{i+j}\det(M_{ij})$.

Example 7.15. Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & -2 & -1 \\ 3 & 3 & 3 \end{bmatrix}.$$

Then we have

$$M_{1,1} = \begin{bmatrix} -2 & -1 \\ 3 & 3 \end{bmatrix} \qquad M_{3,2} = \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}.$$

Fact 7.16 (Cofactor Expansion)**.** *Let A be a $n \times n$ matrix.*

If $A \in M_{1\times 1}$ then $A = \begin{bmatrix} a_{11} \end{bmatrix}$ and $\det A = a_{11}$.

Otherwise, for any k we have

$$\begin{aligned} \det(A) &= \sum_{i=1}^{n} a_{ki}A_{ki} = a_{k1}A_{k1} + a_{k2}A_{k2} + \cdots + a_{kn}A_{kn} \\ &= \sum_{i=1}^{n} a_{ik}A_{ik} = a_{1k}A_{1k} + a_{2k}A_{2k} + \cdots + a_{nk}A_{nk}. \end{aligned}$$

Thus we may compute the determinant of a matrix inductively, using cofactor expansion. We can expand along any row or column; we should pick the one that makes our job easiest.

Remark 7.17. This is usually taken to be the definition of determinant. Feel free to think of it that way, and the fact about eigenvectors as a theorem.

You can also think of the determinant as the unique multilinear map that satisfies certain properties. You probably shouldn't, at the moment. But you can.

Example 7.18. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. If we expand along the last row, we get

$$\begin{aligned} \det A &= 0 \cdot (-1)^{3+1} \det \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix} + 0 \cdot (-1)^{3+2} \det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} + 2 \cdot (-1)^{3+3} \det \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix} \\ &= 2 \det \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix} = 2\left(0 \cdot (-1)^{2+1} \det \begin{bmatrix} 2 \end{bmatrix} + 5 \cdot (-1)^{2+2} \det \begin{bmatrix} 3 \end{bmatrix}\right) \\ &= 2(0 + 5 \cdot 3) = 30. \end{aligned}$$

Example 7.19. Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & -2 & -1 \\ 3 & 3 & 3 \end{bmatrix}.$$

We'd like to expand along the row or column wiht the most zeros, but we don't have any.

I'm going to expand along the bottom row because at least everything is the same.

$$\begin{aligned}
\det A &= 3(-1)^{3+1}\det\begin{bmatrix}1 & 2\\ -2 & -1\end{bmatrix} + 3(-1)^{3+2}\det\begin{bmatrix}3 & 2\\ 5 & -1\end{bmatrix} + 3(-1)^{3+3}\begin{bmatrix}3 & 1\\ 5 & -2\end{bmatrix}\\
&= 3\left(1(-1)^{1+1}(-1) + 2(-1)^{1+2}(-2)\right) - 3\left(3(-1)^{1+1}(-1) + 2(-1)^{1+2}5\right)\\
&\quad + 3\left(3(-1)^{1+1}(-2) + 1(-1)^{1+2}(5)\right)\\
&= 3(-1+4) - 3(-3-10) + 3(-6-5) = 9 + 39 - 33 = 15.
\end{aligned}$$

Using this method, we can compute the derivative of any size of matrix. But for small matrices we can work out quick formulas that encode all this information.

Proposition 7.20.

$$\det\begin{bmatrix}a & b\\ c & d\end{bmatrix} = ad - bc \qquad \det\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix} = aei + bfg + cdh - gec - hfa - idb.$$

7.2.2 Properties of Determinants

We'd like to do things to make computing determinants easier, in addition to the formulas I just gave. We can start by proving some simple results.

Proposition 7.21. *If A is a $n \times n$ triangular matrix, then* $\det A$ *is the product of the diagonal entries of A.*

Proof. We use cofactor expansion; at each step, we have a row or column with only one non-zero entry, on the diagonal. At the end of the cofactor expansion we have simply taken the product of the diagonal entries. □

Proposition 7.22. *If A has a row or column of all zeroes, then* $\det A = 0$.

Proof. Do cofactor expansion along the row of all zeros. □

Proposition 7.23. $\det A^T = \det A$.

Proof. Do a cofactor expansion along the column of A^T that corresponds to the row you expanded along in A, or vice versa. □

Fact 7.24 (Row Operations)**.**

- *Interchanging two rows multiplies the determinant by* -1.

- *Multiplying a row by a scalar multiplies the determinant by that scalar.*
- *Adding a multiple of one row to another row does not change the determinant.*
- $$\det \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{r}_n \end{bmatrix} + \det \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{b}_i \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = \det \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{a}_i + \mathbf{b}_i \\ \vdots \\ \mathbf{r}_n \end{bmatrix}.$$

Proof. The proof is really tedious and just involves a bunch of inductions on cofactor expansions. □

Example 7.25.

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = 1 \qquad \det \begin{bmatrix} 3 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = 3$$

$$\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 \qquad \det \begin{bmatrix} 4 & 4 & 4 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = 3 + 1 = 4.$$

Corollary 7.26. $\det A = 0$ *if and only if the rows of A are linearly dependent.*

Proposition 7.27. *A matrix A is invertible if and only if* $\det A \neq 0$.

Proof. We can view this proof in two different ways.

From the eigenvalue perspective: $\det A$ is the product of the eigenvalues. Thus $\det A = 0$ if and only if 0 is an eigenvalue of A. But 0 is an eigenvalue of A if and only if A has non-trivial kernel, and A is invertible if and only if $\ker(A)$ is trivial.

From the cofactor perspective: if A is invertible it is row-equivalent to the identity matrix, which has determinant 1. None of the row operations can change a determinant from zero to non-zero or vice versa, so $\det A$ is nonzero.

Conversely, if A is not invertible, it is row-equivalent to a matrix with a row of all zeros, which has determinant zero. Since row operations cannot change a determinant from non-zero to zero, $\det A = 0$ as well. □

Fact 7.28. *If A, B are $n \times n$ matrices, then* $\det(AB) = \det(A)\det(B)$.

Corollary 7.29. *If A is a nonsingular matrix, then* $\det(A^{-1}) = \frac{1}{\det A}$.

Remark 7.30. This is why the inverse of a matrix so often has the same denominator appearing in most of the entries; it's the reciprocal of the determinant.

Example 7.31. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

We check this by multiplying the two of them:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

7.3 Characteristic Polynomials

Definition 7.32. We say that $\chi_A(\lambda) = \det(A - \lambda I)$ is the *characteristic polynomial* of A. This is a polynomial in one variable, λ. We call the equation $\chi_A(\lambda) = 0$ the *characteristic equation* of A.

Proposition 7.33. *The real number λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A. That is, the roots of $\chi_A(\lambda)$ is the set of eigenvalues of A.*

Proof. Recall that $\mathbf{v}$ is an eigenvector with eigenvalue λ if and only if $\mathbf{v} \in \ker(A - \lambda I)$. Thus λ is an eigenvalue if and only if $\ker(A - \lambda I)$ has nontrivial kernel, which occurs if and only if $\det(A - \lambda I) = 0$. □

Example 7.34. Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$.

The characteristic equation is

$$\begin{aligned} 0 = \chi_A(\lambda) &= \begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} \\ &= (3-\lambda)(-2-\lambda) - 2 \cdot 3 = -6 - 3\lambda + 2\lambda + \lambda^2 - 6 \\ &= \lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3) \end{aligned}$$

so the eigenvalues are 4 and -3. We compute

$$A - 4I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

so $\ker(A - 4I) = \{\alpha(2, 1)\}$. Thus the eigenspace corresponding to 4 is $E_4 = \operatorname{Span}\{(2, 1)\}$. Similarly,

$$A + 3I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

so $\ker(A+3I) = \{\alpha(-1,3)\}$. Thus the eigenspace $E_{-3} = \operatorname{Span}\{(-1,3)\}$.

Example 7.35. Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$.

The characteristic equation is

$$\begin{aligned} 0 = \chi_A(\lambda) &= \left| \begin{pmatrix} 5-\lambda & 1 \\ 3 & 3-\lambda \end{pmatrix} \right| \\ &= (3-\lambda)(5-\lambda) - 1 \cdot 3 = 15 - 8\lambda + \lambda^2 - 3 \\ &= \lambda^2 - 8\lambda + 12 = (\lambda-6)(\lambda-2) \end{aligned}$$

so the eigenvalues are 6 and 2.

$$A - 6I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

has kernel $\{\alpha(1,1)\}$, so the eigenspace $E_6 = \operatorname{Span}\{(1,1)\}$.

$$A - 2I = \begin{bmatrix} 3 & 1 \\ 3 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

has kernel $\{\alpha(-1,3)\}$, so the eigenspace $E_2 = \operatorname{Span}\{(-1,3)\}$.

Example 7.36. Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$.

The characteristic equation is

$$\begin{aligned} 0 = \chi_A(\lambda) &= \left| \begin{pmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{pmatrix} \right| \\ &= (2-\lambda)(-2-\lambda)(2-\lambda) - 3 - 3 - ((-2-\lambda) - 3(2-\lambda) - 3(2-\lambda)) \\ &= -\lambda^3 + 2\lambda^2 + 4\lambda - 8 - 6 + 2 + \lambda + 12 - 6\lambda \\ &= -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda-1)^2 \end{aligned}$$

so the eigenvalues are 0 and 1 (twice). We have

$$A - 0I = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\ker(A) = \{\alpha(1,1,1)\}$, and $E_0 = \operatorname{Span}\{(1,1,1)\}$. We also have

$$A - I = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\ker(A - I) = \{\alpha(3,1,0) + \beta(-1,0,1)\}$, and $E_1 = \operatorname{Span}\{(3,1,0),(-1,0,1)\}$.

Proposition 7.37. *If A is a $n \times n$ matrix and n is odd, then A has at least one eigenvalue.*

Proof. Recall that a degree n polynomial always has at least one real root if n is odd. Thus if $A \in M_{n\times n}$, $\chi_A(\lambda)$ is degree n, and has a real root, which is an eigenvalue of A. □

Example 7.38. Find the eigenvalues and corresponding eigenspaces of $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Since this matrix is triangluar, we know the eigenvalues are $2, 4, 2$. We solve

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\ker(A - 2I) = \{\alpha(0,0,1)\}$, so $E_2 = \operatorname{Span}\{(0,0,1)\}$. Similarly,

$$A - 4I = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\ker(A - 4I) = \{\alpha(0,1,0)\}$ so $E_4 = \operatorname{Span}\{(0,1,0)\}$.

Notice that in this case, the span of the eigenvectors is only 2-dimensional; the eigenvectors don't span the whole domain.

7.4 Similarity and Trace

Notice that the eigenvalues are a property of the linear transformation, not of the matrix. Thus two matrices representing the same linear transformation should have the same eigenvalues. And indeed, this is the case.

Proposition 7.39. *Suppose A and B are similar $n \times n$ matrices, so there exists U such that $B = U^{-1}AU$. Then:*

- $\det(A) = \det(B)$
- $\chi_A(\lambda) = \chi_B(\lambda)$
- *A and B have the same set of eigenvalues.*

Proof. We can prove these two ways. From a formal perspective, we know that A and B must represent the same linear transformation, and since all of these things are properties of the linear transformation, they must be the same for similar matrices.

From a more concrete algebraic perspective, we have:

- $\det(B) = \det(U^{-1}AU) = \det(U^{-1})\det(A)\det(U) = \frac{1}{\det(U)}\det(A)\det(U) = \det(A)$.
- For any λ, we have
$$U^{-1}(A - \lambda I)U = U^{-1}AU - U^{-1}\lambda IU = B - \lambda I,$$
so we have $(A - \lambda I) \sim (B - \lambda I)$. By the previous result, $\det(A - \lambda I) = \det(B - \lambda I)$.
- The eigenvalues are the roots of the characteristic polynomial. Since $\chi_A(\lambda) = \chi_B(\lambda)$, the roots are the same and so the eigenvalues are the same.

□

Example 7.40. Let $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and let $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, so that

$$B = U^{-1}AU = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}.$$

Clearly $A \sim B$. We can see immediately that $\chi_A(\lambda) = (2-\lambda)(1-\lambda)(1-\lambda) = 2 - 5\lambda + 4\lambda^2 - \lambda^3$ and $\det(A) = 2$. With a little more work, we have

$$\begin{aligned}
\chi_B(\lambda) &= \det\begin{bmatrix} -\lambda & 2 & 0 \\ 2 & 3-\lambda & 3 \\ 1 & -2 & 1-\lambda \end{bmatrix} \\
&= -\lambda\left((3-\lambda)(1-\lambda) - (-2\cdot 3)\right) - 2\left(2(1-\lambda) - 1\cdot 3\right) \\
&= -3\lambda + \lambda^2 + 3\lambda^2 - \lambda^3 - 6\lambda - 4 + 4\lambda + 6 \\
&= 2 - 5\lambda + 4\lambda^2 - \lambda^3 = \chi_A(\lambda).
\end{aligned}$$

Remark 7.41. The converse of this theorem is not true. Similar matrices always have the same characteristic polynomial; but sometimes matrices with the same characteristic polynomial are not similar.

If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then $\chi_A(\lambda) = (1-\lambda)^2 = \chi_I(\lambda)$. But clearly A is not similar to the identity, since $U^{-1}IU = I$, and so the only matrix similar to I is itself.

Since the characteristic polynomials of similar matrices are the same, they clearly have all the same coefficients. In fact, we can see that the determinant is just the constant term of the characteristic polynomial, $\chi_A(0)$. There's one other coefficient that's often important. It's not the highest degree coefficient, which is always ± 1; but the second-highest coefficient is often interesting and useful.

Definition 7.42. If $L : V \to V$ is a linear transformation on a n-dimensional vector space, we define the *trace* of L to be $\operatorname{Tr}(L) = (-1)^{n-1} a_{n-1}$ where $\chi_L(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0$.

If A is a $n \times n$ matrix, we define the *trace* of A to be the trace of the linear transformation represented by A. Thus $\operatorname{Tr}(A) = (-1)^{n-1} a_{n-1}$ where $\chi_A(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$.

Proposition 7.43.

- *If $A \sim B$ then* $\operatorname{Tr}(A) = \operatorname{Tr}(B)$.
- $\operatorname{Tr}(A)$ *is the sum of the eigenvalues of A (weighted by multiplicity).*
- $\operatorname{Tr}(A) = \sum_{i=1}^n a_{ii}$ *is the sum of the entries on the main diagonal of A.*

Proof.

- This follows from the fact that $\chi_A(\lambda) = \chi_B(\lambda)$.
- If $\chi_A(\lambda) = (\lambda - \lambda_1)^{d_1} \dots (\lambda - \lambda_k)^{d_k}$, then when we multiply this out, the $n-1$ coefficient will be $\sum_{i=1}^k d_k \lambda_k$.
- Proof by induction.

□

Remark 7.44. This tells us that the trace is very easy to compute; unlike the determinant, it doesn't depend on any non-diagonal entries, and just requires some fast, simple addition.

This also tells us that the trace is a *similarity invariant*, meaning that similar matrices have the same trace. Thus we can quickly test whether two matrices might be similar by computing the traces of both.

But notice that, like with the determinant, we can have two matrices which are not similar but have the same trace.

If the matrix A is given as a function of T, there is a specific sense in which the trace is related to the derivative of the determinant.

Proposition 7.45. *The trace is a linear multiplicative map on matrices. That is:*

- $\operatorname{Tr}(A+B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)$
- $\operatorname{Tr}(rA) = r \operatorname{Tr}(A)$
- $\operatorname{Tr}(A^T) = \operatorname{Tr}(A)$

Proof. These follow from the characterization of the trace as the sum of the diagonal elements. □

Remark 7.46. In class I said that $\operatorname{Tr}(AB) = \operatorname{Tr}(A)\operatorname{Tr}(B)$. This is actually not true.

Example 7.47. Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 1 \\ 2 & -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 & 1 \\ 4 & -3 & -3 \\ 2 & 1 & 0 \end{bmatrix}$. Then $\operatorname{Tr}(A) = 3+4+2 = 9$ and $\operatorname{Tr}(B) = 5-3+0 = 2$, so we know that $A \not\sim B$.

If $C = \begin{bmatrix} 4 & -2 & 3 \\ 5 & 1 & 7 \\ 1 & 1 & 4 \end{bmatrix}$ then $\operatorname{Tr}(C) = 4+1+4 = 9$, so it's possible that $C \sim A$. But we'd need to do more work to confirm this. On just this evidence, it probably isn't.

In fact we can compute that $\chi_A(\lambda) = -x^3+9x^2-24x+26 \neq -x^3+9x^2-17x-22 = \chi_C(\lambda)$, so the matrices actually aren't similar.

7.5 Diagonalization

Definition 7.48. If D is a $n \times n$ matrix such that $a_{ij} = 0$ whenever $i \neq j$, we say that D is *diagonal*.

Proposition 7.49. *Let* $D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}$ *be a diagonal $n \times n$ matrix. Then:*

- *Each standard basis vector $\mathbf{e}_i$ is an eigenvector of D with eigenvalue d_{ii}.*
- $\det(D) = \prod_{i=1}^n d_{ii}$ *is the product of the diagonal entries.*

- *$\mathbb{R}^n$ is spanned by the eigenvectors of D.*

Proof.
- We have

$$D\mathbf{e}_i = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ d_{ii} \\ \vdots \\ 0 \end{bmatrix} = d_{ii}\mathbf{e}_i.$$

- The determinant is the product of the eigenvalues, which are the diagonal entries.
- The standard basis vectors are eigenvalues, and span $\mathbb{R}^n$.

□

Definition 7.50. We say a linear transformation is *diagonalizable* if its matrix in some basis is diagonal.

We say a matrix is *diagonalizable* if its linear transformation is diagonalizable. Thus A is diagonalizable if A is similar to some diagonal matrix.

Proposition 7.51. *Let A be a $n \times n$ matrix. Then:*

1. *A is diagonalizable if and only if the eigenvectors of A span $\mathbb{R}^n$.*
2. *A is diagonalizable if and only if it has n linearly independent eigenvectors.*
3. *If A has n distinct eigenvalues, then A is diagonalizable.*

Proof.
1. Suppose A is diagonalizable, i.e. there is an invertible matrix U and a diagonal matrix D such that $A = U^{-1}DU$. Let F be the image of the standard basis under U^{-1}; then
$$A\mathbf{f}_i = U^{-1}DU\mathbf{f}_i = U^{-1}D\mathbf{e}_i = U^{-1}d_{ii}\mathbf{e}_i = d_{ii}U^{-1}\mathbf{e}_i = d_{ii}\mathbf{f}_i.$$
Thus $\mathbf{f}_i$ is an eigenvector for each i, so we have a basis of eigenvectors.

   Conversely Suppose the eigenvectors of A span $\mathbb{R}^n$. Then in particular there is a basis $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ of eigenvectors. Let U be the matrix that sends the standard basis to F. Then for each i we have
$$U^{-1}AU\mathbf{e}_i = U^{-1}A\mathbf{f}_i = U^{-1}\lambda_i\mathbf{f}_i = \lambda_i U^{-1}\mathbf{f}_i = \lambda_i\mathbf{e}_i$$
and thus $U^{-1}AU$ is a diagonal matrix with $d_{ii} = \lambda_i$. Thus A is diagonalizable.

2. A set of n linearly independent vectors is a basis for $\mathbb{R}^n$. Thus A has n linearly indpendent eigenvectors if and only if the eigenvectors span $\mathbb{R}^n$.

3. Let $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be a set of eigenvectors corresponding to each eigenvalue. Then this set is linearly independent by proposition 7.8, and thus A has n linearly independent eigenvectors.

□

Remark 7.52. Notice that the converse of (3) is not true. For instance, the identity has only one eigenvalue, but is clearly diagonalizable (and already diagonalized).

Corollary 7.53. *If A is a $n \times n$ matrix and $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ is a basis of eigenvectors, and U is the matrix sending the standard basis to F, then $D = U^{-1}AU$ is a diagonal matrix.*

We say that the matrix U diagonalizes *A.*

Remark 7.54. Diagonalization is not unique; the matrix U depends on the choice of basis. However, since the diagonal entries are the eigenvalues, they will be the same (up to reordering) for any diagonalization.

Example 7.55. Let $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$. We know that the eigenvalues are 4 and -3, so the matrix is diagonalizable; the corresponding eigenvectors are $(2, 1)$ and $(-1, 3)$. So we set

$$\begin{aligned} U &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \\ U^{-1} &= \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \\ U^{-1}AU &= \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 & 3 \\ 4 & -9 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 28 & 0 \\ 0 & -21 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}. \end{aligned}$$

Example 7.56. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. We saw that the eigenvalues are $0, 1, 1$. The eigenvectors are $(1,1,1),(3,1,0),(-1,0,1)$, so we set

$$U = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$U^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$U^{-1}AU = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 7.57. We saw that the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ had eigenspaces $E_2 = \operatorname{Span}\{(0,0,1)\}$ and $E_4 = \operatorname{Span}\{(0,1,0)\}$. The eigenvectors do not span $\mathbb{R}^3$, so A is not diagonalizable.

In general I don't really expect triangular matrices with repeated eigenvalues to be diagonal, but treating this thought fully is beyond the scope of this course.

There are two different major uses for diagonalization. The first is to tell us the basis we "should" be working in, and to allow us to change bases to that basis. The basis in which your operator is diagonal is the basis in which your operator is "really" working; it divides your space up into the dimensions along which your operator really works.

Eigenvectors and diagonalization are often used in various sorts of data analysis. The eigenvector corresponding to the largest eigenvalue is the most significant input, so diagonalization can tell us which components of our data are most important to whatever phenomenon we're studying; this is the idea behind "principal component analysis".

They are also used in various sorts of approximate computations: if your linear operator has eigenvalues of 5, 3, 1, .1, .1, -.1, .0005, you can get a pretty good approximation of your operator by ignoring the eigenvectors corresponding to the small eigenvalues, and only worrying about the large ones.

Second, we can use diagonalization to simplify many matrix computations. We need to make two observations: one about diagonal matrices, the other about similar matrices.

Proposition 7.58. *Suppose C and D are two diagonal matrices with diagonal entries given by c_{ii}, d_{ii} respectively. Then their product is a diagonal matrix given by*

$$\begin{bmatrix} c_{11} & 0 & 0 & \dots & 0 \\ 0 & c_{22} & 0 & \dots & 0 \\ 0 & 0 & c_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 & \dots & 0 \\ 0 & d_{22} & 0 & \dots & 0 \\ 0 & 0 & d_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{nn} \end{bmatrix} = \begin{bmatrix} c_{11}d_{11} & 0 & 0 & \dots & 0 \\ 0 & c_{22}d_{22} & 0 & \dots & 0 \\ 0 & 0 & c_{33}d_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_{nn}d_{nn} \end{bmatrix}.$$

Proposition 7.59. *If $A = U^{-1}BU$, then $A^n = U^{-1}B^nU$.*

Proof.

$$\begin{aligned} A^n = (U^{-1}BU)^n &= U^{-1}BUU^{-1}BU \dots U^{-1}BUU^{-1}BU \\ U^{-1}BI_nB \dots IBIBU &= U^{-1}BB \dots BBU = U^{-1}B^nU. \end{aligned}$$

□

Example 7.60. Let Let $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$. Find A^5.

If $U^{-1}AU = D$, then $UU^{-1}AUU^{-1} = UDU^{-1}$ and thus $A = UDU^{-1}$. So

$$\begin{aligned}
D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} &= \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = U^{-1}AU \\
A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = UDU^{-1} \\
A^5 = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}^5 &= \left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right)^5 \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}^5\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1024 & 0 \\ 0 & -243 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3072 & 1024 \\ 243 & -486 \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} 5901 & 2534 \\ 3801 & -434 \end{bmatrix} = \begin{bmatrix} 843 & 362 \\ 543 & -62 \end{bmatrix}.
\end{aligned}$$

Example 7.61. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. Find a formula for A^n.

We have

$$\begin{aligned}
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \\
\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}^n &= \left(\begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \right)^n \\
&= \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.
\end{aligned}$$

Corollary 7.62. *If A is a diagonalizable matrix whose eigenvalues are only zero or one, then $A^n = A$ for any n.*

Markov chains blah

Example 7.63. $B = \begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix}$ has eigenvalues $1/2, 1$ with eigenvectors $(1, -1)$ and $(2, 3)$.

$$\begin{aligned}
U &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\
U^{-1} &= \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \\
U^{-1}BU &= \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .5 & 2 \\ -.5 & 3 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 2.5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix} \\
B = UDU^{-1} &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \\
B^n &= \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \right)^n \\
&= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix}^n \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \\
\lim_{n\to\infty} B^n &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}.
\end{aligned}$$