Math 214 Spring 2019
Linear Algebra HW 10 Solutions
Due Friday, April 26

For all these problems, justify your answers; do not just write "yes" or "no" or give a single number.

1. Find the orthogonal decomposition of $(2, -1, 5, 6)$ with respect to $U = \text{Span}\{(1, 1, 1, 0), (1, 0, -1, 1)\}$.

   Solution: We see that $(1, 1, 1, 0) \cdot (1, 0, -1, 1) = 1 - 1 = 0$, so this is an orthonormal basis for U. We compute

$$\text{proj}_{(1,1,1,0)} \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} = \frac{(2, -1, 5, 6) \cdot (1, 1, 1, 0)}{(1, 1, 1, 0) \cdot (1, 1, 1, 0)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{proj}_{(1,0,-1,1)} \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} = \frac{(2, -1, 5, 6) \cdot (1, 0, -1, 1)}{(1, 0, -1, 1) \cdot (1, 0, -1, 1)} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix}_U = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix}_{U^\perp} = \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 4 \\ 5 \end{bmatrix}.$$

   We can check that the second vector is in fact in $U^\perp$ by taking the inner product with the two basis vectors for U.

2. Let V be a vector space and $L : V \to V$ a linear transformation, and let λ be a scalar. Prove that the eigenspace corresponding to λ is a subspace of V, using the subspace theorem. (In class we proved this a different way; here I want you to use the subspace theorem specifically).

   Solution:

   (a) $L\mathbf{0} = \mathbf{0} = \lambda\mathbf{0}$, so $\mathbf{0} \in E_\lambda$.

   (b) If $\mathbf{u} \in E_\lambda$ and $r \in \mathbb{R}$, then $L(r\mathbf{u}) = rL(\mathbf{u}) = r\lambda u = \lambda(r\mathbf{u})$, so $r\mathbf{u} \in E_\lambda$.

(c) If $\mathbf{u}, \mathbf{v} \in E_\lambda$ then $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u}+\mathbf{v})$, so $\mathbf{u}+\mathbf{v} \in E_\lambda$.

Thus by the subspace theorem, E_λ is a subspace of V.

3. Which of the following are eigenvectors of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}?$$

What are the corresponding eigenvalues?

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Solution:

$$A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \neq r\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad A\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \neq r\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

4. ($\star$) Let $V = \mathcal{D}(\mathbb{R}, \mathbb{R})$ be the space of differentiable real functions, and consider the linear transformation $\frac{d^2}{dx^2} : V \to V$. Find two linearly independent eigenvectors with eigenvalue 1. Find two linearly independent eigenvectors with eigenvalue -1.

Solution: We can reason about this by noticing that an eigenvector of $\frac{d}{dx}$ with eigenvalue r is also an eigenvector of $\frac{d^2}{dx^2}$ of eigenvalue r^2. So the eigenvectors with eigenvalue 1 are e^{rx} such that $r^2 = 1$; thus the eigenvectors are e^x and e^{-x}.

The eigenvalue of -1 is a bit harder. We want $r^2 = -1$, which has no real solutions. (If we allow complex numbers, e^{ix} and e^{-ix} actually work). But if we think back to calculus, we realize that $\sin(x)$ and $\cos(x)$ are both eigenvectors with eigenvalue -1.

5. Find all eigenvalues and the corresponding eigenvectors for

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

Solution: $\chi_A(\lambda) = x^2 - 7x + 12 = (x-4)(x-3)$ so the eigenvalues are 4 and 3. Row reduction gives us that $(1,1)$ is an eigenvector with eigenvalue 4, and $(3,2)$ is an eigenvector with eigenvalue 3.

$\chi_B(\lambda) = x^2 - 2x + 1 = (x-1)^2$ so the only eigenvalue is 1. Row reduction gives us that $(-1,1)$ is an eigenvector with eigenvalue 1.

6. Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}.$$

Solution: $\chi_A(\lambda) = -x^3 + x^2 + 5x + 3 = -(x+1)^2(x-3)$ so the eigenvalues are 3 and -1. Row reduction gives us that $(2, 3, 2)$ is an eigenvector with eigenvalue 3, and $(-1, 0, 1)$ and $(0, 1, 0)$ are eigenvectors with eigenvalue -1.

7. Find the determinants of the following matrices. You should not need to perform any detailed computations for this problem.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & e & -2 \\ 2 & 2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 2 & 1 & 3 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1/4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 1 & 3 \\ -2 & 0 & -2 \\ 5 & 4 & 1 \end{bmatrix}.$$

Solution: $\det A = 0$ since the third column is a multiple of the first.

$\det B = -21$ since it's a triangular matrix and we can just multiply the diagonal entries.

$\det C = -1$ since swapping a pair of rows makes it into the identity matrix.

$\det D = 0$ since the first column minus the second column is the third column.

8. Find the determinant of the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{bmatrix}.$$

Solution: $\det A = -6 - 5 = -11$.

$\det B = 0 + 12 + 0 - 6 - 0 - 0 = 6$.

$\det C = 0 + 4 - 6 + 6 - 16 - 0 = -12$.

9. Suppose $A, B \in M_{n\times n}$ with $\det(A) = 3$ and $\det(B) = 5$. Find

   (a) $\det(A^{-1})$

   (b) $\det(AB^2)$

   (c) $\det(3B)$

   (d) $\det(B^T A)$.

Solution:

   (a) $1/3$

   (b) 75

   (c) $3^n \cdot 5$ (the answer 15 is wrong, as is any answer that doesn't include n as a parameter).

(d) 15.

10. Find the characteristic polynomial and the eigenvalues with multiplicity of the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: $\chi_A(\lambda) = -x^3 + x^2 + 5x + 3 = -(x+1)^2(x-3)$ so the eigenvalues are 3 with multiplicity 1, and -1 with multiplicity 2.

$\chi_B(\lambda) = x^3 - x^2 = -x^2(x-1)$ so the eigenvalues are 0 with multiplicity 2, and 1 with multiplicity 1.