

Math 214 Spring 2019
 Linear Algebra HW 3 Solutions
 Due Friday, February 15

1. (★) Prove that $\mathcal{F}(\mathbb{R}, \mathbb{R})$, the set of functions from $\mathbb{R} \rightarrow \mathbb{R}$, is a vector space.

Solution: We need to check all ten vector space properties.

- i. We have vector addition defined by $(f + g)(x) = f(x) + g(x)$. This does give a function, so the vector space is closed under addition.
- ii. $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$.
- iii. $((f + g) + h)(x) = (f + g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + (g + h)(x) = (f + (g + h))(x)$.
- iv. Let $\mathbf{0}$ be the zero function defined by $\mathbf{0}(x) = 0$. Then we see that $(f + \mathbf{0})(x) = f(x) + \mathbf{0}(x) = f(x) + 0 = f(x)$.
- v. Define $(-f)(x)$ by $(-f)(x) = -f(x)$. Then $(f + (-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = \mathbf{0}(x)$.
- vi. Define scalar multiplication by $(rf)(x) = rf(x)$. This does give a function, so the vector space is closed under multiplication.
- vii. $(r(f + g))(x) = r(f + g)(x) = r(f(x) + g(x)) = rf(x) + rg(x) = (rf)(x) + (rg)(x)$.
- viii. $((r + s)f)(x) = (r + s)f(x) = rf(x) + sf(x) = (rf)(x) + (sf)(x)$.
- ix. $(r(sf))(x) = r(sf)(x) = rsf(x) = (rs)f(x) = ((rs)f)(x)$.
- x. $(1 \cdot f)(x) = 1f(x) = f(x)$.

Thus $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space.

2. Prove that if $r\mathbf{u} = \mathbf{0}$, then either $r = 0$ or $\mathbf{u} = \mathbf{0}$.

Solution: Suppose $r\mathbf{u} = \mathbf{0}$ and $r \neq 0$. Then $1/r$ is a real number, and we can compute

$$\begin{aligned} \frac{1}{r}(r\mathbf{u}) &= \frac{1}{r}\mathbf{0} \\ \left(\frac{1}{r}\right) \cdot r\mathbf{u} &= \frac{1}{r}\mathbf{0} && \text{multiplicative associativity} \\ 1 \cdot \mathbf{u} &= \frac{1}{r}\mathbf{0} && \text{arithmetic} \\ \mathbf{u} &= \frac{1}{r}\mathbf{0} && \text{multiplicative identity} \\ \mathbf{u} &= \mathbf{0} && \text{proposition from class.} \end{aligned}$$

Thus if $r \neq 0$ then $\mathbf{u} = \mathbf{0}$.

3. (★) Show that the zero vector is unique. That is, if \mathbf{v} is a vector with the property that $\mathbf{v} + \mathbf{u} = \mathbf{u}$ for every vector $\mathbf{u} \in V$, then $\mathbf{v} = \mathbf{0}$.

Solution: Here are two different proofs:

First, we can simply suppose $\mathbf{v} + \mathbf{u} = \mathbf{u}$ for some $\mathbf{v}, \mathbf{u} \in V$. Then

$$\begin{aligned} \mathbf{v} + \mathbf{u} &= \mathbf{u} + \mathbf{0} && \text{additive identity} \\ \mathbf{v} &= \mathbf{0} && \text{cancellation.} \end{aligned}$$

Possibly slightly better is to observe that if $\mathbf{0}$ is not unique, then there is some $\hat{\mathbf{0}}$ with the same property. Then we have

$$\begin{aligned} \mathbf{0} + \hat{\mathbf{0}} &= \hat{\mathbf{0}} && \text{additive identity} \\ \mathbf{0} + \hat{\mathbf{0}} &= \mathbf{0} && \text{given property} \\ \mathbf{0} &= \hat{\mathbf{0}} && \text{transitive property.} \end{aligned}$$

4. (a) Show that the set $\{(x, x, y, y) | x, y \in \mathbb{R}\}$ is a subspace of \mathbb{R}^4 .
(b) Show that the set $\{(x, y, 0) | x, y \in \mathbb{R}\}$ is a subspace of \mathbb{R}^3 . What does this subspace look like geometrically?
(c) Show that the set $\{(x, 2x, 3x) | x \in \mathbb{R}\}$ is a subspace of \mathbb{R}^3 . What does this subspace look like geometrically?

Solution:

- (a) We need to check the three subspace properties.

- i. The zero vector $(0, 0, 0, 0)$ is in this set, corresponding to $x = y = 0$.
- ii. If $(x_1, x_1, y_1, y_1), (x_2, x_2, y_2, y_2)$ are in the set, then

$$(x_1, x_1, y_1, y_1) + (x_2, x_2, y_2, y_2) = (x_1 + x_2, x_1 + x_2, y_1 + y_2, y_1 + y_2)$$

is also in the set (corresponding to $x = x_1 + x_2, y = y_1 + y_2$).

- iii. If $r \in \mathbb{R}$ and (x_1, x_1, y_1, y_1) is in the set, then

$$r(x_1, x_1, y_1, y_1) = (rx_1, rx_1, ry_1, ry_1)$$

is in the set (corresponding to $x = rx_1, y = ry_1$).

Thus it is a subspace by the subspace proposition.

- (b) We check the three subspace properties.

- i. The zero vector $(0, 0, 0)$ is in this set, corresponding to $x = y = 0$.
- ii. If $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ are in the set, then

$$(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0)$$

is in the set, corresponding to $x = x_1 + x_2, y = y_1 + y_2$.

iii. If $r \in \mathbb{R}$ and $(x_1, y_1, 0)$ is a vector in the set, then

$$r(x_1, y_1, 0) = (rx_1, ry_1, 0)$$

is in the set, corresponding to $x = rx_1, y = ry_1$.

This is the xy plane through the origin.

(c) We need to check the three subspace properties.

i. The zero vector $(0, 0, 0)$ is in the set.

ii. If $(x, 2x, 3x)$ and $(y, 2y, 3y)$ are in the set, then

$$(x, 2x, 3x) + (y, 2y, 3y) = (x + y, 2(x + y), 3(x + y))$$

is also in the set.

iii. If $r \in \mathbb{R}$ and $(x, 2x, 3x)$ is a vector in the set, then

$$r(x, 2x, 3x) = (rx, 2(rx), 3(rx))$$

is also a vector in the set.

Thus it is a subspace by the subspace proposition. The set is a line through the origin (angling up).

5. (a) Show that the set $\{f : \mathbb{R} \rightarrow \mathbb{R} \mid f(0) = 0\}$ is a vector space. (Hint: Show it is a subspace of something we know is a vector space).
(b) Show that the set $\{f : \mathbb{R} \rightarrow \mathbb{R} \mid f(0) = 1\}$ is not a vector space.

Solution:

- (a) This is a set of functions, so it is a subset of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. So we just need to check the three subspace properties.
- If $\mathbf{0}$ is the zero function given by $\mathbf{0}(x) = 0$, then $\mathbf{0}(0) = 0$ and thus $\mathbf{0}$ is in the set.
 - Suppose f, g are functions with $f(x) = 0 = g(x)$. Then $(f + g)(x) = f(x) + g(x) = 0 + 0 = 0$, so $f + g$ is in the set.
 - Suppose f is a function with $f(x) = 0$ and $r \in \mathbb{R}$. Then $(rf)(0) = rf(0) = r0 = 0$ so rf is in the set.

Thus the set is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ by the subspace proposition.

- (b) We just need to show that one of the three subspace properties fails. In this case, we see that $\mathbf{0}(0) = 0 \neq 1$, so the zero vector is not contained in this set. Thus it is not a vector space.
6. (a) Show that if n is a positive integer, then the set $\mathcal{P}_n(x)$ of polynomials of degree at most n is a vector space.
(b) Show that the set $\mathcal{C}(\mathbb{R}, \mathbb{R})$ the set of continuous functions of one real variable is a vector space.

Solution:

(a) We know that $\mathcal{P}_n(x) \subseteq \mathcal{P}(x)$ the space of all polynomials, so we just have to check the three subspace properties.

i. The zero vector is the zero polynomial $0 = 0 + 0x + \cdots + 0x^n \in \mathcal{P}_n(x)$.

ii. If $a_0 + a_1x + \cdots + a_nx^n, b_0 + b_1x + \cdots + b_nx^n \in \mathcal{P}_n(x)$, then

$$a_0 + a_1x + \cdots + a_nx^n + b_0 + b_1x + \cdots + b_nx^n \in \mathcal{P}_n(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \in \mathcal{P}_n(x)$$

iii. If $a_0 + a_1x + \cdots + a_nx^n \in \mathcal{P}_n(x)$ then

$$r(a_0 + a_1x + \cdots + a_nx^n) = (ra_0) + (ra_1)x + \cdots + (ra_n)x^n \in \mathcal{P}_n(x).$$

Thus $\mathcal{P}_n(x)$ is a subspace of $\mathcal{P}(x)$ by the subspace proposition.

(b) We know that $\mathcal{C}(\mathbb{R}, \mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$, so we just need to check the three subspace properties.

i. The zero function $f(x) = 0$ is continuous, so $\mathbf{0} \in \mathcal{C}(\mathbb{R}, \mathbb{R})$.

ii. If $f, g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ then f, g are continuous functions, so $f + g$ is a continuous function (from calculus 1) and thus $f + g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$.

iii. If $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $r \in \mathbb{R}$ then f is a continuous function, and thus rf is a continuous function and thus in $\mathcal{C}(\mathbb{R}, \mathbb{R})$.

Thus $\mathcal{C}(\mathbb{R}, \mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

7. Which of the following are vector spaces? You don't need to justify your answers.

(a) $\{f : \mathbb{R} \rightarrow \mathbb{R} | f(0) = 0 \text{ and } f(1) = 0\}$

(b) $\{f : \mathbb{R} \rightarrow \mathbb{R} | f(0) = 0 \text{ or } f(1) = 0\}$

(c) $\{f : \mathbb{R} \rightarrow \mathbb{R} | f \text{ is constant}\}$

(d) $\mathcal{C}([a, b], \mathbb{R})$ the space of continuous functions from $[a, b]$ to \mathbb{R} .

Solution:

(a) yes

(b) no

(c) yes

(d) yes

8. Which of the following are vector spaces? You don't need to justify your answers.

(a) $\{(a, b) \in \mathbb{R}^2 | a + b = 0\}$

(b) $\{(a, b) \in \mathbb{R}^2 | a + b = 3\}$

(c) $\{a_0 + a_1x + a_2x^2 \in \mathcal{P}_2(x) | a_1 = 1\}$.

(d) $\{a_0 + a_2x^2 + a_3x^3 + a_5x^5 \in \mathcal{P}_5(x)\}$.

Solution:

(a) yes

(b) no

(c) no

(d) yes