Math 214 Spring 2019
Linear Algebra HW 5 Solutions
Due Friday, March 8

For all these problems, justify your answers; do not just write "yes" or "no".

1. Is $B = \{(1,2,3), (-2,1,0), (1,0,1)\}$ a basis for $V = \mathbb{R}^3$?

   Solution: We can check spanning:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_3 - 2\alpha_2 \\ 2\alpha_1 + \alpha_2 \\ 3\alpha_1 + \alpha_3 \end{bmatrix}$$

   giving

$$a = \alpha_1 + \alpha_3 - 2\alpha_2 \qquad b = 2\alpha_1 + \alpha_2 \qquad c = 3\alpha_1 + \alpha_3$$

   and thus

$$\begin{aligned} \alpha_2 &= (\alpha_1 + \alpha_3 - a)/2 \\ b &= 2\alpha_1 + (\alpha_1 + \alpha_3 - a)/2 = 5/2\alpha_1 + \alpha_3/2 - a/2 \\ \alpha_3 &= a + 2b - 5\alpha_1 \\ c &= 3\alpha_1 + (a + 2b - 5\alpha_1) = a + 2b - 2\alpha_1 \\ \alpha_1 &= a/2 + b - c/2 \end{aligned}$$

   and thus we have a solution given by

$$\alpha_1 = a/2 + b - c/2 \qquad \alpha_2 = -a - b + c \qquad \alpha_3 = -3a/2 - 3b + 5c/2.$$

   Thus B spans. Since $\dim \mathbb{R}^3 = 3$ and B has three elements, it must be a basis.

   Alternatively we can check linear independence:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_3 - \alpha_2 \\ 2\alpha_1 + \alpha_2 \\ 3\alpha_1 + \alpha_3 \end{bmatrix}$$

   giving

$$0a = \alpha_1 + \alpha_3 - 2\alpha_2 \qquad 0 = 2\alpha_1 + \alpha_2 \qquad 0 = 3\alpha_1 + \alpha_3$$

and thus

$$\begin{aligned}
\alpha_2 &= (\alpha_1 + \alpha_3)/2 \\
0 &= 2\alpha_1 + (\alpha_1 + \alpha_3)/2 = 5/2\alpha_1 + \alpha_3/2 \\
\alpha_3 &= -5\alpha_1 \\
0 &= 3\alpha_1 - 5\alpha_1 = -2\alpha_1 \\
\alpha_1 &= 0
\end{aligned}$$

so that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, showing that B is linearly independent. As before, we can argue that since $\dim \mathbb{R}^3 = 3$ that makes B a basis; or we can check both spanning and linear independence.

2. Is $B = \{(2, 1, 3), (3, -1, 4), (2, 6, 4)\}$ a basis for $V = \mathbb{R}^3$?

Solution: We can check spanning:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 + 3\alpha_2 + 2\alpha_3 \\ \alpha_1 - \alpha_2 + 6\alpha_3 \\ 3\alpha_1 + 4\alpha_2 + 4\alpha_3 \end{bmatrix}$$

giving

$$a = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 \qquad b = \alpha_1 - \alpha_2 + 6\alpha_3 \qquad c = 3\alpha_1 + 4\alpha_2 + 4\alpha_3$$

and thus

$$\begin{aligned}
\alpha_2 &= \alpha_1 + 6\alpha_3 - b \\
a &= 2\alpha_1 + 3(\alpha_1 + 6\alpha_3 - b) + 2\alpha_3 \\
&= 5\alpha_1 + 20\alpha_3 - 3b \\
\alpha_1 &= -4\alpha_3 + 3b/5 + a/5 \\
c &= 3(3b/5 + a/5 - 4\alpha_3) + 4(\alpha_1 + 6\alpha_3 - b) + 4\alpha_3 \\
&= 3a/5 - 11b/5 + 4\alpha_1 + 16\alpha_3 \\
&= 7a/5 + b/5
\end{aligned}$$

And thus every solution satisfies $5c = 7a + b$ and so B does not span, and is not a basis.

Alternatively we could check linear independence. Then we would have

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 + 3\alpha_2 + 2\alpha_3 \\ \alpha_1 - \alpha_2 + 6\alpha_3 \\ 3\alpha_1 + 4\alpha_2 + 4\alpha_3 \end{bmatrix}$$

giving

$$0 = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 \qquad 0 = \alpha_1 - \alpha_2 + 6\alpha_3 \qquad 0 = 3\alpha_1 + 4\alpha_2 + 4\alpha_3$$

and thus

$$\begin{aligned}
\alpha_2 &= \alpha_1 + 6\alpha_3 \\
0 &= 2\alpha_1 + 3(\alpha_1 + 6\alpha_3) + 2\alpha_3 \\
&= 5\alpha_1 + 20\alpha_3 \\
\alpha_1 &= -4\alpha_3 \\
0 &= 3(-4\alpha_3) + 4(\alpha_1 + 6\alpha_3) + 4\alpha_3 \\
&= 4\alpha_1 + 16\alpha_3 \\
&= 0
\end{aligned}$$

and we see that for any α_3 there is a solution; indeed, we have the $\alpha_3 = 1, \alpha_1 = -4, \alpha_2 = 2$ as a nontrivial solution to the linear independence equation, so B is not linearly independent and thus is not a basis.

3. Is $B = \{1 + x, 1 + x^2, 1 + x^3, x + x^2\}$ a basis for $V = \mathcal{P}_3(x)$?

   Solution: We can check either spanning or linear independence. As a spanning spot check, we see we can get $2 = (1 + x) + (1 + x^2) - (x + x^2)$, and from there we can get each of

$$\begin{aligned}
x &= (1 + x) - \frac{1}{2}(2) = (x + 1) - \frac{1}{2}((1 + x) + (1 + x^2) - (x + x^2) \\
&= \frac{1}{2}(1 + x) - \frac{1}{2}(1 + x^2) + \frac{1}{2}(x + x^2) \\
x^2 &= \frac{1}{2}(1 + x^2) - \frac{1}{2}(1 + x) + \frac{1}{2}(x + x^2) \\
x^3 &= (1 + x^3) - \frac{1}{2}(1 + x) - \frac{1}{2}(1 + x^2) + \frac{1}{2}(x + x^2).
\end{aligned}$$

Thus $\{1, x, x^2, x^3\} \subset \text{Span}(B)$ so $\mathcal{P}_3(x) = \text{Span}(\{1, x, x^2, x^3\} \subset \text{Span}(B)$ so B is a spanning set.

We could also prove this the long/direct way, by solving a system. We write

$$\begin{aligned}
a_0 + a_1 x + a_2 x^2 + a_3 x^3 &= b(1 + x) + c(1 + x^2) + d(1 + x^3) + e(x + x^2) \\
&= (b + c + d) + (b + e)x + (c + e)x^2 + dx^3
\end{aligned}$$

which gives

$$a_0 = b + c + d \qquad a_1 = b + e \qquad a_2 = c + e \qquad a_3 = d.$$

We then have

$$\begin{aligned}
d &= a_3 \\
e &= a_2 - c \\
b &= a_1 - a_2 + c \\
c &= a_0 - b - d = a_0 - (a_1 - a_2 + c) - a_3 \\
&= a_0 - a_1 + a_2 - a_3 - c
\end{aligned}$$

which gives

$$b = a_0/2 + a_1/2 - a_2/2 - a_3/2 \qquad c = a_0/2 - a_1/2 + a_2/2 - a_3/2$$
$$d = a_3 \qquad e = -a_0/2 + a_1/2 + a_2/2 + a_3/2$$

and thus the set spans. Since it is the right size, it must be a basis and linearly independent.

Alternatively, we can check linear independence. A spot check doesn't show us any way to write one element as a linear combination of the others, so we guess it is independent. We write

$$\begin{aligned} 0 &= b(1+x) + c(1+x^2) + d(1+x^3) + e(x+x^2) \\ &= (b+c+d) + (b+e)x + (c+e)x^2 + dx^3 \end{aligned}$$

which gives

$$= b+c+d \qquad 0 = b+e \qquad 0 = c+e \qquad 0 = d.$$

We then have

$$\begin{aligned} d &= 0 \\ e &= -c \\ b &= c \\ c &= -b-d = -c-0 \\ 2c &= 0 \end{aligned}$$

and thus we have $c = 0, b = 0, e = 0, d = 0$ and the system is indeed linearly independent. Since it has the correct number of elements it must be a basis.

4. Let $V = \{(a,b,c) : a+b = c\}$. Find a basis for V.

   Solution: There are many correct answers here. I intended you to use the philosphy of the padding lemma. First find a (non-zero) vector that is in this space; an easy choice is $(1,0,1)$. Now we simply need to find another vector that is not dependent on the first vector; an easy choice here is $(0,1,1)$.

   We check that this set is linearly independent: suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}.$$

   Then we have $a = 0, b = 0, a+b = 0$, which holds only when $a = b = 0$, so the set is linearly independent.

   We check that it spans: if

$$\begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$$

   then we have $x = a, y = b, x+y = a+b$ which is solved by $a = x$ and $b = y$. Thus the set spans, and so is a basis.

5. Let $S = \{(1,2,0),(3,2,-1)\}$. Find a set $B \supseteq S$ that is a basis for $\mathbb{R}^3$.

Solution: First we check that S is linearly independent: suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} a+3b \\ 2a+2b \\ -b \end{bmatrix}$$

so we have

$$a + 3b = 0 \qquad 2a + 2b = 0 \qquad -b = 0.$$

Then $b = 0$ and so $a = 0$, so S is linearly independent.

Now by padding, we just need to find a vector that's not in the span of S. We guess $(1,0,0)$ is not in the span; indeed solving

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} a+3b \\ 2a+2b \\ -b \end{bmatrix}$$

so we have

$$a + 3b = 1 \qquad 2a + 2b = 0 \qquad -b = 0.$$

The third equation gives $b = 0$, then the second equation gives $a = 0$, so we have $0 = 1$ a contradiction.

Thus $(1,0,0)$ is not in the span of S, and so $B = \{(1,2,0),(3,2,-1),(1,0,0)\}$ is a basis for $\mathbb{R}^3$.

6. Let $T = \{(1,2,2),(2,5,4),(1,3,2),(2,7,4),(1,1,0)\}$. Find a set $B \subseteq T$ that is a basis for $\mathbb{R}^3$.

Solution: First we should check that this spans. The easy way is to observe

$$\begin{aligned}
(1,0,0) &= (1,2,2) + (1,1,0) - (1,3,2) \\
(0,1,0) &= (1,3,2) - (1,2,2) \\
(0,0,1) &= \frac{1}{2}(1,3,2) - \frac{1}{2}(1,0,0) - \frac{3}{2}(0,1,0).
\end{aligned}$$

We can also set up and solve a system of linear equations. We see that this indeed spans.

By basis reduction we just need to find vectors that depend linearly on the other vectors and remove them. We have $(2,5,4) = (1,2,2) + (1,3,2)$, so we can remove $(2,5,4)$. Then we have $(2,7,4) = 3(1,3,2) - (1,2,2)$ so we can remove $(2,7,4)$. Thus we have $B = \{(1,2,2),(1,3,2),(1,1,0)\}$ is a basis for $\mathbb{R}^3$.

7. What is the dimension of $\operatorname{Span}\{\sin^2, \cos^2, 1\} \subset \mathcal{F}(\mathbb{R},\mathbb{R})$?

8. ($\star$) Suppose V is a finite–dimensional vector space and U is a subspace of V. If $\dim U = \dim V$, prove that $U = V$.

Solution: Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for U. Then B is a linearly independent subset of V, and has a number of element equal to $\dim V$, and thus B is a basis.

Longer form: if B does not span V, then by the padding lemma there is some $\mathbf{u} \in V$ so that $B \cup \{\mathbf{u}\} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{u}\}$ is linearly independent in V, and we can continue padding until we have a basis. But every basis for V has n elements, and this basis would have more than n elements, a contradiction. Thus B must already span V, and thus be a basis.