Math 214 Spring 2019
Linear Algebra HW 6
Due Friday, March 22

For all these problems, justify your answers; do not just write "yes" or "no".

1. Let $L : U \to V$ be a linear transformation. Prove that $\ker(L)$ is a subspace of U.

2. ($\star$) Let $\mathcal{C}([a,b],\mathbb{R})$ be the space of continuous functions defined on the closed interval $[a,b]$. Prove that the function from $\mathcal{C}([a,b],\mathbb{R})$ to $\mathbb{R}$ given by $f \mapsto \int_a^b f(t)\,dt$ is a linear transformation. (Be careful: what is a vector in the domain? What is a vector in the codomain/image?)

   What is the kernel of this transformation?

3. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$L((x,y,z)) = \begin{bmatrix} x+y+z \\ 3x-2y+z \\ 2z \end{bmatrix}.$$

   (a) Prove that L is a linear transformation.

   (b) Find a basis for the kernel.

4. Let $\mathcal{F}(\mathbb{R},\mathbb{R})$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Define $E_0 : \mathcal{F}(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ to be the function given by $E_0(f) = f(0)$. (Thus if $f(x) = x^3 + 3x + 1$ then $E_0(f) = f(0) = 1$).

   (a) Prove that E_0 is a linear transformation.

   (b) What is the kernel of E_0? What is the image?

5. Let $P_Z : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection map onto the xy plane, given by $P(x,y,z) = (x,y,0)$.

   (a) Prove that P_z is a linear transformation.

   (b) Find a basis for the kernel of P_z.

   (c) Find a matrix for P_z with respect to the standard basis.

   (d) Prove that $P_z(P_z(\mathbf{u})) = P_z(\mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}^3$. Linear transformations with this property are called *projections* and we will revisit them later. (They are also sometimes called *idempotent* if you're feeling particularly fancy).

6. (a) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If $L((1,2)) = (-2,3)$ and $L((1,-1)) = (5,2)$, what is $L((7,5))$?

   (b) $E = \{(1,1,0),(1,0,1),(0,1,1)\}$ is a basis for $\mathbb{R}^3$. Take the vector $\mathbf{u}$ represented by $(2,3,4)$ in the standard basis, and calculate $[\mathbf{u}]_E$.

   (c) $F = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ is a basis for $\mathcal{P}_3(x)$.
   Calculate $[3+5x-2x^2+x^3]_F$.

7. Let $L : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Prove that there is some real number $r \in \mathbb{R}$ such that $L(x) = rx$ for all $x \in \mathbb{R}$. (In other words, any linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is given by multiplication by a scalar).

8. ($\star$) Let $L : U \to V$ be a linear transformation, and let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for U. Prove that $L(U) = \mathrm{Span}(L(E))$. That is, prove the image of L is just the span of the image of E under L.