Math 214 Spring 2019
Linear Algebra HW 6 Solutions
Due Friday, March 22

For all these problems, justify your answers; do not just write "yes" or "no".

1. Let $L : U \to V$ be a linear transformation. Prove that $\ker(L)$ is a subspace of U.

   Solution: We need to check three things for the subspace theorem.

   (a) We see that $L(\mathbf{0}) = L(0 \cdot \mathbf{0}) = 0L(\mathbf{0}) = \mathbf{0}$, so $\mathbf{0} \in \ker(L)$.

   (b) Suppose $r \in \mathbb{R}, \mathbf{u} \in \ker(L)$. Then $L(r\mathbf{u}) = rL(\mathbf{u}) = r\mathbf{0} = \mathbf{0}$, so $r\mathbf{u} \in \ker(L)$.

   (c) Suppose $\mathbf{u}, \mathbf{v} \in \ker(L)$. Then $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u})+L(\mathbf{v}) = \mathbf{0}+\mathbf{0} = \mathbf{0}$, so $\mathbf{u}+\in \ker(L)$.

   Thus by the subspace theorem, $\ker(L)$ is a subspace of U.

2. ($\star$) Let $\mathcal{C}([a,b],\mathbb{R})$ be the space of continuous functions defined on the closed interval $[a,b]$. Prove that the function from $\mathcal{C}([a,b],\mathbb{R})$ to $\mathbb{R}$ given by $f \mapsto \int_a^b f(t)\,dt$ is a linear transformation. (Be careful: what is a vector in the domain? What is a vector in the codomain/image?)

   What is the kernel of this transformation?

   Solution: If $r \in \mathbb{R}$ and $f \in \mathcal{C}([a,b],\mathbb{R})$, then

$$\int_a^b (rf)(t)\,dt = \int_a^b rf(t)\,dt = r\int_a^b f(t)\,dt$$

   and if $f, g \in \mathcal{C}([a,b],\mathbb{R})$, then

$$\int_a^b (f+g)(t)\,dt = \int_a^b f(t) + g(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.$$

   Thus by definition this map is a linear transformation.

   The kernel is the set of all functions f such that $\int_a^b f(t)\,dt = 0$. We can think of this as the set of all functions whose average value over the interval $[a,b]$ is zero.

3. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$L((x,y,z)) = \begin{bmatrix} x+y+z \\ 3x-2y+z \\ 2z \end{bmatrix}.$$

(a) Prove that L is a linear transformation.

(b) Find a basis for the kernel.

Solution:

(a) We have

$$L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+y_1+y_2+z_1+z_2 \\ 3x_1+3x_2-2y_1-2y_2+z_1+z_2 \\ 2z_1+2z_2 \end{bmatrix}$$
$$= \begin{bmatrix} x_1+y_1+z_1 \\ 3x_1-2y_1+z_1 \\ 2z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2+z_2 \\ 3x_2-2y_2+z_2 \\ 2z_2 \end{bmatrix}$$
$$= L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + \left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$

(b) This transformation has the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ which we can row-reduce to get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The nullspace of this matrix is the trivial vector space $\{\mathbf{0}\}$, so we have $\ker(L) = \{\mathbf{0}\}$ and a basis for the kernel of L is $\{\}$.

4. Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Define $E_0 : \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ to be the function given by $E_0(f) = f(0)$. (Thus if $f(x) = x^3 + 3x + 1$ then $E_0(f) = f(0) = 1$).

(a) Prove that E_0 is a linear transformation.

(b) What is the kernel of E_0? What is the image?

Solution:

(a) We see that $E_0(f + g) = (f + g)(0) = f(0) + g(0) = E_0(f) + E_0(g)$. We also see that $E_0(rf) = (rf)(0) = rf(0) = rE_0(f)$. Thus E_0 is a linear transformation by definition.

(b) $E_0(f) = 0$ if and only if $f(0) = 0$. Thus the kernel of E_0 is $\{f : \mathbb{R} \to \mathbb{R} | f(0) = 0\}$. The image of E_0 is all reals, since for any r we can take the constant function $f(x) = r$ and we have $E_0(f) = f(0) = r$. (Thus E_0 is onto, but not one-to-one).

5. Let $P_Z : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection map onto the xy plane, given by $P(x, y, z) = (x, y, 0)$.

(a) Prove that P_z is a linear transformation.

(b) Find a basis for the kernel of P_z.

(c) Find a matrix for P_z with respect to the standard basis.

(d) Prove that $P_z(P_z(\mathbf{u})) = P_z(\mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}^3$. Linear transformations with this property are called *projections* and we will revisit them later. (They are also sometimes called *idempotent* if you're feeling particularly fancy).

Solution:

(a) $P(r(x,y,z)) = P(rx,ry,rz) = (rx,ry,0) = r(x,y,0) = rP(x,y,z)$.
$P((x_1,y_1,z_1)+(x_2,y_2,z_2)) = P(x_1+x_2, y_1+y_2, z_1+z_2) = (x_1+x_2, y_1+y_2, 0) = (x_1,y_1,0)+(x_2,y_2,0) = P(x_1,y_1,z_1)+P(x_2,y_2,z_2)$. Thus by defnition P_z is a linear transformation.

(b) $P_z(x,y,z) = (x,y,0) = \mathbf{0}$ when $x = y = 0$. Thus $\ker(P_z) = \{(0,0,z)\}$. Thus a basis for the kernel is $\{(0,0,1)\}$.

(c) We have $P_z(1,0,0) = (1,0,0), P_z(0,1,0) = (0,1,0), P_z(0,0,1) = (0,0,0)$. Thus a matrix for P_z is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(d) $P_z(P_z(x,y,z)) = P_z(x,y,0) = (x,y,0) = P_z(x,y,z)$.

6. (a) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If $L((1,2)) = (-2,3)$ and $L((1,-1)) = (5,2)$, what is $L((7,5))$?

(b) $E = \{(1,1,0),(1,0,1),(0,1,1)\}$ is a basis for $\mathbb{R}^3$. Take the vector $\mathbf{u}$ represented by $(2,3,4)$ in the standard basis, and calculate $[\mathbf{u}]_E$.

(c) $F = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ is a basis for $\mathcal{P}_3(x)$.
Calculate $[3+5x-2x^2+x^3]_F$.

7. Let $L : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Prove that there is some real number $r \in \mathbb{R}$ such that $L(x) = rx$ for all $x \in \mathbb{R}$. (In other words, any linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is given by multiplication by a scalar).

Solution: Set $r = L(1)$. Then if $x \in \mathbb{R}$, we have $L(x) = xL(1) = xr = rx$.

Alternatively, you can observe that a linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is given by multiplication by a 1×1 matrix, which is just a scalar.

8. ($\star$) Let $L : U \to V$ be a linear transformation, and let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for U. Prove that $L(U) = \operatorname{Span}(L(E))$. That is, prove the image of L is just the span of the image of E under L.

Solution:

The first is mutual subset inclusion. Suppose $\mathbf{v} \in \operatorname{Span}(L(E))$. Then we can write

$$\begin{aligned} \mathbf{v} &= a_1 L(\mathbf{e}_1) + \cdots + a_n L(\mathbf{e}_n) = L(a\mathbf{e}_1) + \cdots + L(a\mathbf{e}_n) \\ &= L(a\mathbf{e}_1 + \cdots + a\mathbf{e}_n) \in L(U). \end{aligned}$$

Thus $\operatorname{Span}(L(E)) \subseteq L(U)$.

Conversely, let $\mathbf{v} \in L(U)$. Then we can write $\mathbf{v} = L(\mathbf{u})$ for some $\mathbf{u} \in U$. Since E is a basis for U, we can write

$$\begin{aligned} \mathbf{u} &= a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n \\ L(\mathbf{u}) &= L(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) = L(a_1\mathbf{e}_1) + \cdots + L(a_n\mathbf{e}_n) \\ &= a_1L(\mathbf{e}_1) + \cdots + a_nL(\mathbf{e}_n) \in \operatorname{Span}(L(E)). \end{aligned}$$

Thus $L(U) \subseteq \operatorname{Span}(L(E))$, so $L(U) = \operatorname{Span}(L(E))$.