Math 214 Spring 2019
Linear Algebra HW 7 Solutions
Due Friday, March 29

For all these problems, justify your answers; do not just write "yes" or "no".

1. Let $A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$. Find bases for the rowspace, columnspace, and nullspace of A.

   Solution:

$$\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 14 & 1 & -4 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

   Then a basis for the rowspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -10/7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2/7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

   A basis for the columnspace is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} \right\}.$$

   The nullspace is $\{(10\alpha/7, 2\alpha/7, 0, \alpha)\}$ so a basis for the nullscpace is

$$\left\{ \begin{bmatrix} 10 \\ 2 \\ 0 \\ 7 \end{bmatrix}. \right\}$$

2. Let $B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$. Find bases for the rowspace, columnspace, and nullspace of B.

Solution:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus a basis for the rowspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \right\}$$

A basis for the columnspace is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \right\}.$$

To find the kernel, we set the third column as the free variable, and the kernel is $\{(-2\alpha, 0, \alpha)\}$. Thus a basis is $\{(-2, 0, 1)\}$.

3. Use Gaussian elimination to find a basis for the span of $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$.

Solution: The simplest approach is to make each of these vectors a *row* of a matrix, and then do row reduction.

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 3 & -2 & 5 \\ 2 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a basis for the span.

Alternatively, you could make these vectors the columns of a matrix, and find a basis for the columnspace.

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -4 & 5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the first and third columns have leading 1s, the first and third vectors form a basis for the span. Thus our basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}.$$

4. For each of the following systems of equations, is there a solution? You don't need to find the solution if it exists, but justify your answer. (Hint: think about the column space).

   (a)

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}?$$

   Solution: It's easy to see that $(1,1,1)$ is not in the span of the column vectors, since anything in the span of the column vectors has a middle coordinate of 0. So the system has no solutions.

   (b)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 1 \end{bmatrix}?$$

   Solution: The columns of the matrix form a basis for $\mathbb{R}^3$, so $(3, 17, 1)$ is in their span. Thus there is a solution to this system.

5. (a) Find a basis for the image of the linear transformation $L(x,y,z) = \begin{bmatrix} x+y+z \\ 3x-2y+z \\ 2z \end{bmatrix}$ that we saw in Homework 6 problem 3.

   (b) Find a basis for the image of the operator $P_Z(x,y,z) = (x,y,0)$ that we saw in Homework 6 Problem 5.

   Solution:

   (a) The matrix of this transformation is $\begin{bmatrix} 1 & 1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ which row-reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ as we saw in homework 6. So a basis for the image is

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

   (b) The matrix of this transformation is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which is already row-reduced. So we see a basis for the image is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

6. ($\star$) Let $U = \mathcal{P}_3(x)$, and define a linear map $D : U \to U$ by $D(f(x)) = f'(x)$. Let $E = \{1, x, x^2, x^3\}$ be a basis for U.

   (a) Describe the kernel and image of D.

(b) Find a matrix for D with respect to E (as a basis for both the domain and the codomain).

(c) Find bases for the kernel and image of D.

Solution:

(a) $\ker(D)$ is the set of all polynomials whose derivative is zero, which is just the set of constants. Thus $\ker(D) = \{a_0 + 0x + 0x^2 + 0x^3 : a_0 \in \mathbb{R}\} = \{a_0 : a_0 \in \mathbb{R}\}$.
The image of D is the set of all polynomials of degree 2 or less. If $a_0 + a_1x + a_2x^2$ is a degree two polynomial, then $a_0x + a_1/2x^2 + a_2/3x^3 \in \mathcal{P}_3(x)$ with $D(a_0x + a_1/2x^2 + a_2/3x^3) = a_0 + a_1x + a_2x^2$.

(b) We have

$$[D(1)]_E = [0]_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad [D(x)]_E = [1]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[D(x^2)]_E = [2x]_E = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad [D(x^3)]_E = [3x^2]_E = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

Thus the matrix with respect to E is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) The nullspace of the matrix is $\{(\alpha, 0, 0, 0) | \alpha \in \mathbb{R}\}$, so a basis for the nullspace is $\{(1, 0, 0, 0)\}$. Moving from coordinates back into the original space gives a basis of $\{1\}$.
The columnspace of the matrix is spanned by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

so this is a basis for the columnspace. Moving back to our original space, this corresponds to $\{x, x^2, x^3\}$.

7. ($\star$) Let $L : \mathcal{P}_3(x) \to \mathbb{R}^2$ be given by $L(f(x)) = (f(1), f(2))$, and let $E = \{1, x, x^2, x^3\}$ and $F = \{(1, 0), (0, 1)\}$.

(a) Prove that L is a linear transformation.

(b) Find a matrix with respect to the bases E and F.

(c) Find bases for the kernel and image of L.

Solution: This is a linear transformation, since

$$\begin{aligned} L(rf(x)) &= (rf(1), rf(2)) = r(f(1), f(2)) = rL(f(x)) \\ L((f+g(x)) &= ((f+g)(1), (f+g)(2)) = (f(1)+g(1), f(2)+g(2)) \\ &= (f(1), f(2)) + (g(1), g(2)) = L(f(x)) + L(g(x)). \end{aligned}$$

To compute the matrix, we have

$$\begin{aligned} L(1) &= (1,1) \\ L(x) &= (1,2) \\ L(x^2) &= (1,4) \\ L(x^3) &= (1,8) \\ A &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}. \end{aligned}$$

Row reducing this matrix gives

$$\begin{bmatrix} 1 & 0 & -2 & -6 \\ 0 & 1 & 3 & 7 \end{bmatrix}$$

so the kernel is given by $\{(2\alpha + 6\beta, -3\alpha - 7\beta, \alpha, \beta)\} = \{\alpha(2, -3, 1, 0) + \beta(6, -7, 0, 1)\}$ in E, and thus a basis is $\{2 - 3x + x^2, 6 - 7x + x^3\}$.

The image has basis $\{(1,1), (1,2)\}$ and thus is all of $\mathbb{R}^2$.