Math 214 Spring 2019
Linear Algebra HW 8
Due Friday, April 12

For all these problems, justify your answers; do not just write "yes" or "no".

1. Let E be the standard basis for $\mathbb{R}^3$, and let $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

   (a) Find the transition matrix corresponding to the change of basis from E to F.

   (b) For each of the following vectors (expressed in the standard basis), find the coordinates with respect to F: $(3, 2, 5)$; $(1, 1, 2)$; $(2, 3, 2)$.

2. Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \left\{ \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$, and let $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ be two bases for $\mathbb{R}^3$.

   (a) Find the transition matrix from E to F.

   (b) If $\mathbf{x} = 2\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_3$, find the coordinates of $\mathbf{x}$ with respect to F.

3. Let
$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x - y - z \\ -x + 2y - z \\ -x - y + 2z \end{bmatrix}.$$
   Let A be the matrix of L with respect to the standard basis, and let B be the matrix of L with respect to the basis $F = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$.

   (a) Calculate B, the matrix of L with respect to F directly.

   (b) Calculuate B by finding the matrix U corresponding to a change of basis from F to the standard basis, and calculating $U^{-1}AU$.

4. ($\star$) Let $T : \mathcal{P}_3(x) \to \mathcal{P}_3(x)$ be defined by $L(f(x)) = xf'(x) + f''(x)$.

   (a) Find the matrix A representing T with respect to $E = \{1, x, x^2\}$.

   (b) Find the matrix B representing T with respect to $F\{1, x, 1 + x^2\}$.

   (c) Find the matrix S such that $B = S^{-1}AS$.

(d) If $p(x) = a_0 + a_1x + a_2(1 + x^2)$, calculate $T^n(p(x)) = T(T(\ldots(T(p(x)))))$.

5. Let $\mathbf{u} = (2, 1, 3)$ and $\mathbf{v} = (6, 3, 9)$.
   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.
   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.
   (c) Verify that $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$.

6. Let $\mathbf{u} = (2, -5, 4)$ and $\mathbf{v} = (1, 2, -1)$.
   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.
   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.
   (c) Verify that $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$.

7. Let $\mathbf{u} = (4, 1)$ and $\mathbf{v} = (3, 2)$.
   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.
   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.
   (c) Verify that $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$.

8. Let $\mathbf{u} = (3, 5)$ and $\mathbf{v} = (1, 1)$.
   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.
   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.
   (c) Verify that $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$.