Math 214 Spring 2019
Linear Algebra HW 8 Solutions
Due Friday, April 12

For all these problems, justify your answers; do not just write "yes" or "no".

1. Let E be the standard basis for $\mathbb{R}^3$, and let $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

   (a) Find the transition matrix corresponding to the change of basis from E to F.

   (b) For each of the following vectors (expressed in the standard basis), find the coordinates with respect to F: $(3, 2, 5)$; $(1, 1, 2)$; $(2, 3, 2)$.

Solution:

(a) We have

$$U = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

is the transition matrix from F to E. To find the inverse of this matrix we calculuate

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

so the inverse is

$$U^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

This is the change of basis matrix from E to F.

(b)

$$U^{-1}\begin{bmatrix}3\\2\\5\end{bmatrix} = \begin{bmatrix}2 & 0 & -1\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}3\\2\\5\end{bmatrix} = \begin{bmatrix}1\\-4\\3\end{bmatrix}$$

$$U^{-1}\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}2 & 0 & -1\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$U^{-1}\begin{bmatrix}2\\3\\2\end{bmatrix} = \begin{bmatrix}2 & 0 & -1\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}2\\3\\2\end{bmatrix} = \begin{bmatrix}2\\2\\-1\end{bmatrix}.$$

2. Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \left\{\begin{bmatrix}4\\6\\7\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$, and let $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$ be two bases for $\mathbb{R}^3$.

(a) Find the transition matrix from E to F.

(b) If $\mathbf{x} = 2\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_3$, find the coordinates of $\mathbf{x}$ with respect to F.

Solution:

(a) The transition matrix from E to the standard basis is

$$A = \begin{bmatrix}4 & 0 & 0\\6 & 1 & 1\\7 & 1 & 2\end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$B = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 2\\0 & 0 & 1\end{bmatrix}.$$

The transition matrix from E to F is given by $B^{-1}A$. To compute B^{-1} we write

$$\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\1 & 2 & 2 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\0 & 1 & 1 & -1 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 2 & -1 & 0\\0 & 1 & 1 & -1 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 2 & -1 & 0\\0 & 1 & 0 & -1 & 1 & -1\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right]$$

so

$$B^{-1}A = \begin{bmatrix}2 & -1 & 0\\-1 & 1 & -1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}4 & 0 & 0\\6 & 1 & 1\\7 & 1 & 2\end{bmatrix} = \begin{bmatrix}2 & -1 & -1\\-5 & 0 & -1\\7 & 1 & 2\end{bmatrix}.$$

(b)

$$[\mathbf{x}]_F = B^{-1}A[\mathbf{x}]_E = \begin{bmatrix} 2 & -1 & -1 \\ -5 & 0 & -1 \\ 7 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 9 \end{bmatrix}.$$

We can check our work by computing

$$\mathbf{x} = 2\begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix}$$

$$5\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 6\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix} = \mathbf{x}.$$

3. Let

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y - z \\ -x + 2y - z \\ -x - y + 2z \end{bmatrix}.$$

Let A be the matrix of L with respect to the standard basis, and let B be the matrix of L with respect to the basis $F = \{(1,1,0), (1,0,1), (0,1,1)\}$.

(a) Calculate B, the matrix of L with respect to F directly.

(b) Calculuate B by finding the matrix U corresponding to a change of basis from F to the standard basis, and calculating $U^{-1}AU$.

Solution:

(a)

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

So the matrix of L with respect to F is

$$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(b) We have the matrix of L with respect to the standard basis is

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

and the transition matrix from F to the standard basis is

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

We compute

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 1 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

so

$$U^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Then we have

$$\begin{aligned} B = U^{-1}AU &= \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \end{aligned}$$

This matches our earlier computation for B.

4. ($\star$) Let $T : \mathcal{P}_2(x) \to \mathcal{P}_2(x)$ be defined by $L(f(x)) = xf'(x) + f''(x)$.

   (a) Find the matrix A representing T with respect to $E = \{1, x, x^2\}$.

   (b) Find the matrix B representing T with respect to $F\{1, x, 1 + x^2\}$.

   (c) Find the matrix S such that $B = S^{-1}AS$.

   (d) If $p(x) = a_0 + a_1x + a_2(1 + x^2)$, calculate $T^n(p(x)) = T(T(\ldots(T(p(x)))))$.

Solution:

(a)

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b)

$$\begin{aligned}
T(1) &= 0 \to (0,0,0)\\
T(x) &= x \to (0,1,0)\\
T(1+x^2) &= 2+2x^2 = 2(1+x^2) \to (0,0,2)\\
B &= \begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2 \end{bmatrix}
\end{aligned}$$

(c) S here is the change of basis matrix from F to E. So we expect

$$S = \begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}$$

and indeed we have

$$\begin{aligned}
S^{-1} &= \begin{bmatrix} 1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}\\
S^{-1}AS &= \begin{bmatrix} 1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2\\ 0 & 1 & 0\\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2\\ 0 & 1 & 0\\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2 \end{bmatrix}.
\end{aligned}$$

(d) $[p]_F = (a_0, a_1, a_2)$, so

$$\begin{aligned}
[T^n(p(x))]_F = B^n[p]_F &= \begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2 \end{bmatrix}^n \begin{bmatrix} a_0\\ a_1\\ a_2 \end{bmatrix}\\
&= \begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2^n \end{bmatrix} \begin{bmatrix} a_0\\ a_1\\ a_2 \end{bmatrix} = \begin{bmatrix} 0\\ a_1\\ 2^n a_2 \end{bmatrix}.
\end{aligned}$$

Thus

$$T^n(p(x)) = a_1 x + 2^n a_2(1+x^2) = 2^n a_2 + a_1 x + 2^n a_2 x^2.$$

5. Let $\mathbf{u} = (2,1,3)$ and $\mathbf{v} = (6,3,9)$.

   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

   (c) Verify that $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

Solution:

(a)

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= 42\\
\|\mathbf{u}\| &= \sqrt{14}\\
\|\mathbf{v}\| &= 3\sqrt{14}\\
\cos\theta &= \frac{42}{3\cdot 14} = 1\\
\theta &= \pi/2.
\end{aligned}$$

(b)

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{42}{126}\mathbf{v} = \frac{1}{3}\begin{bmatrix}6\\3\\9\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix}.$$

(c) $\operatorname{Proj}_{\mathbf{v}}\mathbf{u} = \mathbf{u}$ so $u - \operatorname{Proj}_{\mathbf{v}}\mathbf{u} = \mathbf{0}$ is orthogonal to everything, including $\mathbf{u}$. In particular, we compute

$$\left(\begin{bmatrix}2\\1\\3\end{bmatrix} - \begin{bmatrix}2\\1\\3\end{bmatrix}\right)\cdot\begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}2\\1\\3\end{bmatrix} = 0.$$

6. Let $\mathbf{u} = (2, -5, 4)$ and $\mathbf{v} = (1, 2, -1)$.

   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

   (c) Verify that $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{u} - \operatorname{proj}_{\mathbf{v}}\mathbf{u}$.

   Solution:

   (a) We compute

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= -12\\
\|\mathbf{u}\| &= \sqrt{45}\\
\|\mathbf{v}\| &= \sqrt{6}\\
\cos\theta &= \frac{-12}{\sqrt{45}\sqrt{6}} = \approx -.73\\
\theta &\approx 2.39.
\end{aligned}$$

   (b)

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}}\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\\
&= \frac{-12}{6}\mathbf{v} = \begin{bmatrix}-2\\-4\\2\end{bmatrix}
\end{aligned}$$

(c) We compute that

$$\begin{aligned}
\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \\
\operatorname{proj}_{\mathbf{v}} \mathbf{u} \cdot (\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}) &= \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = 0.
\end{aligned}$$

7. Let $\mathbf{u} = (4, 1)$ and $\mathbf{v} = (3, 2)$.

   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

   (c) Verify that $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

   Solution:

   (a)

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= 14 \\
\|\mathbf{u}\| &= \sqrt{17} \\
\|\mathbf{v}\| &= \sqrt{13} \\
\cos\theta &= \frac{14}{\sqrt{17 \cdot 13}} \approx .94 \\
\theta &\approx .34.
\end{aligned}$$

   (b)

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{14}{13} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

   (c) We compute

$$\begin{aligned}
\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \frac{14}{13} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 10/13 \\ -15/13 \end{bmatrix} \\
\operatorname{proj}_{\mathbf{v}} \mathbf{u} \cdot (\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}) &= \frac{14}{13}(30/13 - 30/13) = 0.
\end{aligned}$$

8. Let $\mathbf{u} = (3, 5)$ and $\mathbf{v} = (1, 1)$.

   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

   (c) Verify that $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

   Solution:

(a)

$$
\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= 8 \\
\|\mathbf{u}\| &= \sqrt{34} \\
\|\mathbf{v}\| &= \sqrt{2} \\
\cos\theta &= \frac{8}{2\sqrt{17}} = \frac{4\sqrt{17}}{17} \approx .97 \\
\theta &= .25.
\end{aligned}
$$

(b)

$$
\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{8}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.
$$

(c)

$$
\begin{aligned}
\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \begin{bmatrix} 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
\operatorname{proj}_{\mathbf{v}} \mathbf{u}(\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}) &= \begin{bmatrix} 4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -4 + 4 = 0.
\end{aligned}
$$