Math 214 Spring 2019
Linear Algebra HW 9
Due Friday, April 19

For all these problems, justify your answers; do not just write "yes" or "no" or give a single number.

1. Let $V = \mathcal{P}_n(x)$ and fix real numbers $x_0, x_1, \ldots, x_n$ be distinct real numbers. For $f, g \in V$, define
$$\langle f, g\rangle = \sum_{i=0}^{n} f(x_i)g(x_i).$$
Prove this is an inner product on V.

   (Hint: See partial proof from class)

2. Let $w_1, \ldots, w_n$ be positive real numbers. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define
$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n} x_i y_i w_i.$$
Prove that this is an inner product on $\mathbb{R}^n$. (The w_i are called the *weights* of the inner product).

3. Let $V = \mathcal{C}([1,3], \mathbb{R})$, with the usual inner product $\langle f, g\rangle = \int_1^3 f(t)g(t)\,dt$. Find $\|1\|$ and $\|x\|$. Find the projection of $1 + x$ onto 1 and x.

4. Prove the Pythagorean law: if $\mathbf{u}, \mathbf{v}$ are orthogonal, then $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$.

5. Let $\mathbf{u}, \mathbf{v}$ be vectors in an inner product space V, with $\mathbf{v} \neq 0$. Let $\mathbf{p} = \operatorname{proj}_{\mathbf{v}} \mathbf{u}$. (Hint: compare the end of section 6.1).

   Prove that $\langle \mathbf{u} - \mathbf{p}, \mathbf{p}\rangle = 0$.

6. Let $V = \mathcal{C}([-\pi, \pi], \mathbb{R})$ with the usual inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt$. Show that $\{1, \sin(x), \cos(x)\}$ is an orthogonal set. Is it orthonormal?

7. Let $V = \mathbb{R}^4$ with the dot product, and let $U = \operatorname{Span}(\{(5,3,1,0), (2,4,3,5), (1,1,1,1)\})$. Use the Gram-Schmidt process to find an orthonormal basis for U.

8. Let $V = \mathcal{P}_2(x)$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$.

   Following the Gram-Schmidt process, convert $\{1, x, x^2\}$ into an orthonormal basis.

9. Let $V = \mathbb{R}^4$ and let $U = \text{Span}(\{(3, 5, 2, 1), (5, 1, -1, -5)\}$. Find an orthonormal basis for $U^{\perp}$.

10. ($\star$) Let $\mathbf{u}_1, \mathbf{u}_2$ form an orthonormal basis for $\mathbb{R}^2$, and suppose $\mathbf{v}$ is a unit vector. If $\mathbf{v} \cdot \mathbf{u}_1 = 1/2$, compute $|\mathbf{v} \cdot \mathbf{u}_2|$. (Hint: the Pythagorean law you prove in number 4 will be helpful here).