Math 214 Spring 2019
Linear Algebra HW 9 Solutions
Due Friday, April 19

For all these problems, justify your answers; do not just write "yes" or "no" or give a single number.

1. Let $V = \mathcal{P}_n(x)$ and fix real numbers $x_0, x_1, \ldots, x_n$ be distinct real numbers. For $f, g \in V$, define
$$\langle f, g \rangle = \sum_{i=0}^{n} f(x_i)g(x_i).$$
Prove this is an inner product on V.

   (Hint: See partial proof from class)

   Solution: We need to prove that this has three properties.

   (a) Positive Definite: We showed this in class. If f is a real polynomial, then $\langle f, f \rangle = \sum_{i=0}^{n} f(x_i)^2 \geq 0$ since it is a sum of non-negative numbers. If $\langle f, f \rangle = 0$ then $f(x_i)^2 = 0$ for each x_i, so f has $n+1$ roots. Since f is a degree n polynomial with $n+1$ roots, f is zero.

   (b) Symmetric: We have
$$\langle f, g \rangle = \sum_{i=0}^{n} f(x_i)g(x_i) = \sum_{i=0}^{n} g(x_i)f(x_i) = \langle g, f \rangle.$$

   (c) Bilinear (only need to check in the first term because the function is symmetric): we see that
$$\begin{aligned}\langle \alpha f + \beta g, h \rangle &= \sum_{i=0}^{n} (\alpha f(x_i) + \beta g(x_i))h(x_i) \\ &= \alpha \sum_{i=0}^{n} f(x_i)h(x_i) + \beta \sum_{i=0}^{n} g(x_i)h(x_i) = \alpha\langle f, h \rangle + \beta\langle g, h \rangle.\end{aligned}$$

2. Let $w_1, \ldots, w_n$ be positive real numbers. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define
$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i w_i.$$
Prove that this is an inner product on $\mathbb{R}^n$. (The w_i are called the *weights* of the inner product).

   Solution: We again need to check three things.

(a) Positive Definite: We have $\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^{n} x_i^2 w_i \geq 0$ since $x_i^2 \geq 0$ and $w_i > 0$. Further, if $\sum_{i=1}^{n} x_i^2 w_i = 0$, then we must have $x_i^2 w_i = 0$ for each i, and since $w_i > 0$ this implies that $x_i^2 = 0$ so $x_i = 0$. Thus $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and is equal to zero if and only if $\mathbf{x} = \mathbf{0}$.

(b) Symmetric:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i w_i = \sum_{i=1}^{n} y_i x_i w_i = \langle \mathbf{y}, \mathbf{x} \rangle.$$

(c) Bilinear:

$$\begin{aligned} \langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle &= \sum_{i=1}^{n} (\alpha x_i + \beta y_i) z_i w_i \\ &= \alpha \sum_{i=1}^{n} x_i z_i w_i + \beta \sum_{i=1}^{n} y_i z_i w_i = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle. \end{aligned}$$

3. Let $V = \mathcal{C}([1,3], \mathbb{R})$, with the usual inner product $\langle f, g \rangle = \int_1^3 f(t)g(t)\, dt$. Find $\|1\|$ and $\|x\|$. Find the projection of $1 + x$ onto 1 and x.

Solution:

$$\begin{aligned} \|1\| &= \sqrt{\int_1^3 1^2\, dx} = \sqrt{x|_1^3} = \sqrt{2} \\ \|x\| &= \sqrt{\int_1^3 x^2\, dx} = \sqrt{x^3/3|_1^3} = \sqrt{26/3} \\ \text{proj}_1\, 1 + x &= \frac{\langle 1 + x, 1 \rangle}{\langle 1, 1 \rangle} 1 = \frac{1}{2} \int_1^3 1 + x\, dx (1) \\ &= \frac{1}{2} (x + x^2/2)|_1^3 (1) = 3(1) \\ \text{proj}_x\, 1 + x &= \frac{\langle 1 + x, x \rangle}{\langle x, x} x = \frac{3}{26} \int_1^3 x + x^2\, dx (x) \\ &= \frac{3}{26} (x^2/2 + x^3/3)|_1^3 (x) = \frac{3}{26} (4 + 26/3) x = \frac{38}{26} x. \end{aligned}$$

4. Prove the Pythagorean law: if $\mathbf{u}, \mathbf{v}$ are orthogonal, then $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$.

Solution:

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + 0 + 0 + \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2. \end{aligned}$$

5. Let $\mathbf{u}, \mathbf{v}$ be vectors in an inner product space V, with $\mathbf{v} \neq 0$. Let $\mathbf{p} = \text{proj}_{\mathbf{v}}\, \mathbf{u}$. (Hint: compare the end of section 6.1).

Prove that $\langle \mathbf{u} - \mathbf{p}, \mathbf{p} \rangle = 0$.

Solution: We calculate

$$\begin{aligned}
\mathbf{p} &= \operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} \\
\langle \mathbf{u} - \mathbf{p}, \mathbf{p} \rangle &= \left\langle \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} \right\rangle \\
&= \left\langle \mathbf{u}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} \right\rangle - \left\langle \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} \right\rangle \\
&= \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \langle \mathbf{u}, \mathbf{v} \rangle - \left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \right)^2 \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle} = 0.
\end{aligned}$$

6. Let $V = \mathcal{C}([-\pi, \pi], \mathbb{R})$ with the usual inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\, dt$. Show that $\{1, \sin(x), \cos(x)\}$ is an orthogonal set. Is it orthonormal?

   Solution:

$$\begin{aligned}
\langle 1, \sin(x) \rangle &= \int_{-\pi}^{\pi} 1 \sin(x)\, dx = -\cos(x)|_{-\pi}^{\pi} = -\cos(\pi) + \cos(-\pi) = 1 - 1 = 0. \\
\langle 1, \cos(x) \rangle &= \int_{-\pi}^{\pi} 1 \cos(x)\, dx = \sin(x)|_{-\pi}^{\pi} = \sin(\pi) - \sin(-\pi) = 0 - 0 = 0. \\
\langle \sin(x), \cos(x) \rangle &= \int_{-\pi}^{\pi} \sin(x) \cos(x)\, dx = \frac{1}{2} \sin^2(x)|_{-\pi}^{\pi} = \frac{1}{2} \left(\sin^2(\pi) - \sin^2(-\pi) \right) \\
&= \frac{1}{2}(0 - 0) = 0.
\end{aligned}$$

   We can see this isn't an orthonormal set because e.g. $\langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1\, dx = 2\pi \neq 1$. If we want an orthonormal set we can just divide each vector by its magnitude.

7. Let $V = \mathbb{R}^4$ with the dot product, and let $U = \operatorname{Span}(\{(5, 3, 1, 0), (2, 4, 3, 5), (1, 1, 1, 1)\})$. Use the Gram-Schmidt process to find an orthonormal basis for U.

   Solution: We can pick any element to start with. It would probably be simplest to start with $(1, 1, 1, 1)$, but I'm going to start with $(5, 3, 1, 0)$ here because that's the order I wrote them in. Starting with any one is fine.

$$\mathbf{f}_1 = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_1 = \frac{1}{\sqrt{25 + 9 + 1}} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{35}} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{f}_2 = \mathbf{e}_2 - \operatorname{proj}_{\mathbf{f}_1} \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix} - \frac{(5,3,1,0)\cdot(2,4,3,5)}{(5,3,1,0)\cdot(5,3,1,0)} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix} - \frac{25}{35} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -11/7 \\ 13/7 \\ 16/7 \\ 5 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -11 \\ 13 \\ 16 \\ 35 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{11^2 + 13^2 + 16^2 + 35^2}} \begin{bmatrix} -11 \\ 13 \\ 16 \\ 35 \end{bmatrix} = \frac{1}{\sqrt{1771}} \begin{bmatrix} -11 \\ 13 \\ 16 \\ 35 \end{bmatrix}$$

$$\mathbf{f}_3 = \mathbf{e}_3 - \operatorname{proj}_{\mathbf{f}_1} \mathbf{e}_3 - \operatorname{proj}_{\mathbf{f}_2} \mathbf{e}_3$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(5,3,1,0)\cdot(1,1,1,1)}{(5,3,1,0)\cdot(5,3,1,0)} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} - \frac{(-11,13,16,35)\cdot(1,1,1,1)}{(-11,13,16,35)\cdot(-11,13,16,35)} \begin{bmatrix} -11 \\ 13 \\ 16 \\ 35 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{9}{35} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} - \frac{53}{1771} \begin{bmatrix} -11 \\ 13 \\ 16 \\ 35 \end{bmatrix} = \frac{1}{1265} \begin{bmatrix} 55 \\ -203 \\ 334 \\ -60 \end{bmatrix}$$

$$\mathbf{u}_3 = \frac{1}{\sqrt{55^2 + 203^2 + 334^2 + 60^2}} \begin{bmatrix} 55 \\ -203 \\ 334 \\ -60 \end{bmatrix} = \frac{1}{3\sqrt{17710}} \begin{bmatrix} 55 \\ -203 \\ 334 \\ -60 \end{bmatrix}.$$

We should check that these are orthogonal; we see that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \frac{1}{\sqrt{35 \cdot 1771}}(5 \cdot (-11) + 3 \cdot 13 + 1 \cdot 16) = \frac{1}{\sqrt{35 \cdot 1771}}(-55 + 39 + 16) = 0$$

$$\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \frac{1}{3\sqrt{35 \cdot 17710}}(5 \cdot 55 + 3 \cdot (-203) + 1 \cdot 334)$$

$$= \frac{1}{3\sqrt{35 \cdot 17710}}(275 - 609 + 334) = 0$$

$$\langle \mathbf{u}_2, \mathbf{u}_3 \rangle = \frac{1}{3\sqrt{1771 \cdot 17710}}((-11)55 + 13(-203) + 16 \cdot 334 + 35(-60))$$

$$= \frac{1}{3 \cdot 1771\sqrt{10}}(-605 - 2639 + 5344 - 2100) = 0.$$

8. Let $V = \mathcal{P}_2(x)$ with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$.

   Following the Gram-Schmidt process, convert $\{1, x, x^2\}$ into an orthonormal basis.

   Solution: We have

$$\mathbf{f}_1 = 1 \qquad \mathbf{u}_1 = \frac{1}{\sqrt{\int_{-1}^{1} 1\,dx}} 1 = \frac{1}{\sqrt{2}}.$$

$$\mathbf{f}_2 = \mathbf{e}_2 - \operatorname{proj}_{\mathbf{f}_1} \mathbf{e}_2 = x - \frac{\int_{-1}^1 x\,dx}{\int_{-1}^1 1\,dx} x = x - \frac{0}{2} x = x$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{\int_{-1}^1 x^2\,dx}} x = \frac{1}{\sqrt{2/3}} x = \sqrt{3/2}x$$

$$\begin{aligned}
\mathbf{f}_3 &= \mathbf{e}_3 - \operatorname{proj}_{\mathbf{f}_1} \mathbf{e}_3 - \operatorname{proj}_{\mathbf{f}_2} \mathbf{e}_3 \\
&= x^2 - \frac{\int_{-1}^1 x^2\,dx}{\int_{-1}^1 1\,dx} 1 - \frac{\int_{-1}^1 x^3\,dx}{\int_{-1}^1 x^2} x \\
&= x^2 - \frac{2/3}{2} 1 - \frac{0}{2/3} x = x^2 - 1/3
\end{aligned}$$

$$\begin{aligned}
\mathbf{u}_3 &= \frac{1}{\sqrt{\int_{-1}^1 (x^2-1/3)^2\,dx}} (x^2 - 1/3) = \frac{1}{\sqrt{8/45}} (x^2 - 1/3) \\
&= \frac{3\sqrt{5}}{2\sqrt{2}} (x^2 - 1/3) = \frac{\sqrt{5}}{2\sqrt{2}} (3x^2 - 1).
\end{aligned}$$

We should check that our basis is in fact orthogonal and orthonormal. We have

$$\begin{aligned}
\langle \mathbf{u}_1, \mathbf{u}_2 \rangle &= \int_{-1}^1 \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{2}} x\,dx = \frac{\sqrt{3}}{2} \int_{-1}^1 x\,dx = 0 \\
\langle \mathbf{u}_1, \mathbf{u}_3 \rangle &= \int_{-1}^1 \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{5}}{2\sqrt{2}} (3x^2 - 1)\,dx = \frac{\sqrt{5}}{4} \int_{-1}^1 3x^2 - 1 \\
&= \frac{\sqrt{5}}{4} (1 - 1 - 1 + 1) = 0. \\
\langle \mathbf{u}_2, \mathbf{u}_3 \rangle &= \int_{-1}^1 \frac{\sqrt{3}}{\sqrt{2}} x \cdot \frac{\sqrt{5}}{2\sqrt{2}} (3x^2 - 1)\,dx = \frac{\sqrt{15}}{4} \int_{-1}^1 3x^3 - x\,dx \\
&= \frac{\sqrt{15}}{4} \left(\frac{3}{4} - \frac{1}{3} - \frac{3}{4} + \frac{1}{3} \right) = 0
\end{aligned}$$

9. Let $V = \mathbb{R}^4$ and let $U = \operatorname{Span}(\{(3,5,2,1),(5,1,-1,-5)\}$. Find an orthonormal basis for $U^\perp$.

Solution: We first need to find a basis for $U^\perp$ at all. $U^\perp$ is the kernel of the matrix whose rows are a basis for U. We write

$$A = \begin{bmatrix} 3 & 5 & 2 & 1 \\ 5 & 1 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 22 & 0 & -7 & -26 \\ 0 & 22 & 13 & 20 \end{bmatrix}$$

which gives us $\{(7,-13,22,0),(26,-20,0,22)\}$ as a basis for $U^\perp$. We just need to orthonormalize.

$$\mathbf{f}_1 = \begin{bmatrix} 7 \\ -13 \\ 22 \\ 0 \end{bmatrix} \qquad \mathbf{u}_1 = \frac{1}{\sqrt{7^2 + 13^2 + 22^2}} \begin{bmatrix} 7 \\ -13 \\ 22 \\ 0 \end{bmatrix} = \frac{1}{702} \begin{bmatrix} 7 \\ -13 \\ 22 \\ 0 \end{bmatrix}$$

$$\mathbf{f}_2 = \mathbf{e}_2 - \text{proj}_{\mathbf{f}_1} \mathbf{e}_2 = \begin{bmatrix} 26 \\ -20 \\ 0 \\ 22 \end{bmatrix} - \frac{(26, -20, 0, 22) \cdot (7, -13, 22, 0)}{(7, -13, 22, 0) \cdot (7, -13, 22, 0)} \begin{bmatrix} 7 \\ -13 \\ 22 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 26 \\ -20 \\ 0 \\ 22 \end{bmatrix} - \frac{442}{702} \begin{bmatrix} 7 \\ -13 \\ 22 \\ 0 \end{bmatrix} = \frac{1}{27} \begin{bmatrix} 583 \\ -319 \\ -374 \\ 594. \end{bmatrix}$$

We should check that these two vectors are orthogonal to each other, and are in the kernel of A. Fortunately they are, so this is an orthonormal basis for $U^\perp$.

10. ($\star$) Let $\mathbf{u}_1, \mathbf{u}_2$ form an orthonormal basis for $\mathbb{R}^2$, and suppose $\mathbf{v}$ is a unit vector. If $\mathbf{v} \cdot \mathbf{u}_1 = 1/2$, compute $|\mathbf{v} \cdot \mathbf{u}_2|$. (Hint: the Pythagorean law you prove in number 4 will be helpful here).

    Solution: Let $\mathbf{v} \cdot \mathbf{u}_2 = r$. Then we have $\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2 = \frac{1}{2}\mathbf{u}_1 + r\mathbf{u}_2$.

    Since $\mathbf{u}_1, \mathbf{u}_2$ are orthogonal unit vectors, we know that

$$\begin{aligned} \|\mathbf{v}\|^2 &= \left\| \frac{1}{2}\mathbf{u}_1 + r\mathbf{u}_2 \right\|^2 = \left\| \frac{1}{2}\mathbf{u}_1 \right\|^2 + \|r\mathbf{u}_2\|^2 \\ 1^2 &= \left(\frac{1}{2} \right)^2 + r^2 \\ r^2 &= 1 - \frac{1}{2} \\ |r| &= \sqrt{3/4} = \frac{\sqrt{3}}{2}. \end{aligned}$$