

# Math 214 Test 2

## Practice Problems

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This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I will update you when I finalize the test format.

**For each of the following sets, check:**

- Does it span the (implicitly given) vector space?
- Is it linearly independent?
- Is it a basis?

1.  $S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$

2.  $S = \{(2, 1, -2), (3, 2, -2), (2, 2, 0)\}$

3.  $S = \{(1, 0, 0, 1), (0, 1, 0, 0), (2, 3, 0, 2)\}$

4.  $S = \{(1, 5, 2), (3, 1, 4), (-1, 3, 7), (2, 8, 1)\}$

5.  $S = \{1 + x^2, 1 + x^3, x - x^2, 5 + x^2 - 4x^3\}$

6.  $S = \{1 + 2x, x + 2x^2, x^2 + 2x^3, 2 + x^3\}$

**Bases**

1. Find a basis for  $\mathbb{R}^3$  containing  $(-1, 3, 2)$  and  $(5, 4, 1)$ .
2. Find a basis for  $\mathbb{R}^3$  containing  $(7, 1, -3)$  and  $(1, 1, 1)$ .
3. Find a basis for  $\mathbb{R}^4$  containing  $(1, 2, 3, 4)$ ,  $(1, 1, 1, 1)$ , and  $(0, 0, 1, 1)$ .
4. Find a basis for  $\mathcal{P}_3(x)$  containing  $1 + 3x^3$ ,  $x^2 - x$ ,  $6 - 2x$ .
5. Find a basis for  $\mathbb{R}^3$  that is a subset of  $\{(1, 1, 1), (2, 4, 6), (7, -1, 2), (2, 5, -2), (3, -6, 4)\}$ .
6. Find a basis for  $\mathbb{R}^2$  that is a subset of  $\{(1, 3), (2, 4), (1, 1)\}$ .
7. Find a basis for  $\mathbb{R}^2$  that is a subset of  $\{(-1, 4), (7, -2), (3, 6)\}$ .
8. Find a basis for  $\mathcal{P}_2(x)$  that is a subset of  $\{1 + x, 3 + x^2, 4 + 3x + 2x^2, x^2 - 7x\}$ .

**For each of the following matrices, find:**

- (a) The reduced row echelon form.
- (b) A basis for the row space.
- (c) A basis for the column space.
- (d) The rank.
- (e) A basis for the null space.
- (f) The nullity.

1. 
$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

2. 
$$\begin{bmatrix} -2 & 4 & 1 \\ -5 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix}$$

3. 
$$\begin{bmatrix} 6 & 2 & 3 & 1 \\ 1 & 5 & 2 & -2 \\ 4 & -4 & 1 & 3 \end{bmatrix}$$

4. 
$$\begin{bmatrix} -1 & 3 & 4 \\ 2 & 5 & 2 \\ 0 & 1 & 3 \\ 4 & 1 & -2 \end{bmatrix}$$

**Find the rank and nullity of the following matrices**

You shouldn't need to do any actual computations here. (Hint: Rank-Nullity theorem).

1. 
$$\begin{bmatrix} 1 & 1 & 1 & 32 & 217 & 53 - e & 3^3 \\ 0 & 1 & 0 & 512 & 256 & 128 & 64 \\ 0 & 1 & 1 & 12345 & 4^{4^4} & 2 & 0 \end{bmatrix}$$

2. 
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

3. 
$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 1 & 3 & 2 & 4 \end{bmatrix}$$

4. 
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the inverses of the following matrices, or show they are not invertible.

1.  $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

2.  $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

3.  $\begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & -1 \\ 1 & -4 & 3 \end{bmatrix}$

4.  $\begin{bmatrix} 2 & 1 & 3 \\ 5 & 0 & 2 \\ 7 & 1 & 5 \end{bmatrix}$

For each of the following functions

- Identify the domain and codomain.
- Determine whether it is a linear transformation.
- Prove your answer from part (b).
- If it is, find a matrix with respect to the standard basis of  $\mathbb{R}^n$ .
- If it is linear, find the kernel and image.

- $f(x, y, z) = (3x^2, x + y, 2z - y)$ .
- $f(x, y, z) = (5x + y, z - 3x, y + z)$ .
- $f(x, y, z) = (x + y, z + y)$ .
- $f(x, y) = (x + y, x - y, 1)$ .

For each of the following functions

- Determine whether it is a linear transformation.
- Prove your answer from part (a).
- If it is, find a matrix with respect to the given bases.
- If it is linear, find the kernel and image.

- $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by  $L(x, y) = (x, y, x + y)$ , with respect to  $E = \{(1, 1), (1, -1)\}$  and  $F = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ .
- $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by  $L(x, y, z) = (x + y + z, x - y)$ , with respect to  $E = \{(1, 1, 1), (1, -1, 0), (0, 0, 1)\}$  and  $F = \{(3, 0), (0, 2)\}$ .
- $L : \mathcal{P}_3(x) \rightarrow \mathbb{R}^2$  given by  $L(f(x)) = (f(1), f(2))$ , with  $E = \{1, x, x^2, x^3\}$  and  $F = \{(1, 0), (0, 1)\}$ .
- $L : \mathcal{P}_2(x) \rightarrow \mathcal{P}_3(x)$  given by  $L(f(x)) = \int_0^x f(t) dt$ , with  $E = \{1, x, x^2\}$  and  $F = \{1, x, x^2, x^3\}$ .
- $L : \mathbb{R}^3 \rightarrow \mathcal{P}_3(x)$  given by  $L(a, b, c) = (a + b + c) + ax + bx^2 + cx^3$ , with  $E = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$  and  $F = \{1, 1 + x, 1 + x^2, 1 + x^3\}$ .
- The function  $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by rotating 90 degrees counterclockwise around the  $z$  axis, and then 135 degrees counterclockwise around the  $x$  axis.

## Isomorphisms

1. Let  $f(x, y, z) = (2x - y, y + z, z + x)$  and  $g(a, b, c) = (a + b - c, a + 2b - 2c, -a - b + 2c)$ . Prove that  $g$  is the inverse of  $f$ .
2. Let  $L(x, y, z) = (x - y, 3x + z, y - 2z)$ . Find a formula for  $L^{-1}$ . (Do *not* leave your answer as a matrix).
3. Let  $T : \mathbb{R}^3 \rightarrow \mathcal{P}_2(x)$  be given by  $T(a, b, c) = (a - c) + (b - c)x + (a + b + c)x^2$ . Find  $T^{-1}$ . (Do *not* leave your answer as a matrix).
4. ( $\star$ ) If  $f(x) \in \mathcal{P}_2(x)$  such that  $f(1) = 4, f(3) = 7, f(4) = 1$ , find  $f(x)$ . (Hint: define an evaluation map from  $\mathcal{P}_2(x)$  to  $\mathbb{R}^3$ ).
5. (a) Suppose  $L : \mathbb{R}^3 \rightarrow \mathcal{P}_2(x)$  is linear and surjective. Prove it is an isomorphism.  
(b) Suppose  $T : \mathcal{P}_5(x) \rightarrow \mathbb{R}^6$  is linear with trivial kernel. Prove it is an isomorphism.

## Proofs

1. Let  $L : U \rightarrow V$  be a linear transformation of vector spaces. Prove that  $L$  is one-to-one if and only if  $\ker(L) = \{\mathbf{0}\}$ .
2. Prove that inverses are unique. That is, let  $f : U \rightarrow V$ , and let  $g, h : V \rightarrow U$  such that  $g(f(\mathbf{u})) = \mathbf{u} = h(f(\mathbf{u}))$  and  $f(g(\mathbf{v})) = \mathbf{v} = f(h(\mathbf{v}))$ . Prove that  $g = h$ .
3. Let  $L : U \rightarrow V$  be a linear transformation.
  - (a) If  $\dim U > \dim V$ , prove that  $L$  is not injective. (There is “too much” in  $U$  to fit it all in  $V$  without repeating).
  - (b) If  $\dim U < \dim V$ , prove that  $L$  is not surjective. (There is “not enough” in  $U$  to cover all of  $V$ ).
  - (c) Find counterexamples to the converses of these statements. That is, find a function  $L : U \rightarrow V$  where  $L$  is not injective, but  $\dim U < \dim V$ . And find a function  $L : U \rightarrow V$  where  $L$  is not surjective, but  $\dim U > \dim V$ .
4. Let  $L : U \rightarrow V$  be a linear transformation of vector spaces, with bases  $E$  and  $F$ . Let  $A$  be the matrix of  $L$  with respect to  $E$  and  $F$ . Prove that  $\mathbf{v} \in \ker(L)$  if and only if  $[\mathbf{v}]_E$  is in the nullspace of  $A$ .
5. Let  $L : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  such that  $\ker(L) = \{(x, y, z, w) : x = y, z = w\}$ . Prove that the image of  $L$  is  $\mathbb{R}^2$ .
6. Let  $A \in M_{m \times n}$  and  $\mathbf{b} \in \mathbb{R}^m$ . Prove that if  $N(A) = \{\mathbf{0}\}$  then the equation  $A\mathbf{x} = \mathbf{b}$  has at most one solution.
7. Let  $U, V$  be 2-dimensional subspaces of  $\mathbb{R}^3$ . On your first test you showed that the set  $U \cap V = \{\mathbf{u} : \mathbf{u} \in U, \mathbf{u} \in V\}$  of vectors in both  $U$  and  $V$  is a subspace of  $\mathbb{R}^3$ . Prove that  $\dim(U \cap V) \neq 0$ . (Hint: let  $\{\mathbf{u}_1, \mathbf{u}_2\}$  be a basis for  $U$  and  $\{\mathbf{v}_1, \mathbf{v}_2\}$  be a basis for  $V$ . What can you say about  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$ ?)