Math 214 Final Exam
Practice Problems

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This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I will update you when I finalize the test format.

Proofs

1. Let Q be the subspace of $\mathcal{P}(x)$ consisting of polynomials with zero constant term. Prove that the function $D : Q \to \mathcal{P}(x)$ given by the derivative is an isomorphism.
2. Let $U = \operatorname{span}\{x, \sin(x), \cos(x), x^5, 1\}$. Find an isomorphism between U and $\mathbb{R}^5$.
3. Suppose V is a vector space and $L : V \to \mathbb{R}^5$ is surjective and $\dim \ker(L) = 2$. What can you say about V?
4. Suppose $T : \mathbb{R}^5 \to \mathcal{P}_4(x)$ and $\dim \ker(T) = 1$. What can you say about $T(\mathbb{R}^5)$?
5. If λ is an eigenvalue of A then prove that λ^{-1} is an eigenvalue of A^{-1}.
6. Suppose $S, T : V \to V$ are linear and have the property that $S(T(\mathbf{v})) = T(S(\mathbf{v}))$ for every $\mathbf{v} \in V$. If $\mathbf{v}$ is an eigenvector of T, prove that $S(\mathbf{v})$ is also an eigenvector of T.
7. Suppose $L : V \to V$ is a linear transformation of rank k. Prove that L has at most $k + 1$ distinct eigenvalues.

Things to Ponder

1. Find a 4×4 matrix with no real eigenvalues. Is it possible to find a 3×3 matrix with no real eigenvalues?
2. Find matrices $A, B \in M_{n\times n}$ such that $\operatorname{Tr}(A)\operatorname{Tr}(B) \neq \operatorname{Tr}(AB)$.
   Find a matrix A such that $\operatorname{Tr}(A^2) < 0$.
3. What happens if you use the Gram-Schmidt process on a set of vectors that isn't linearly independent?

Find the transition matrices between the following bases

1. The standard basis and

$$F = \left\{ \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} \right\}$$

2. The standard basis and

$$F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

3.

$$E = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \quad \text{and} \quad F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

4.

$$E = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Write the given element in the given basis

1. Write $(3, 1, 4)$ in the basis $F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

2. Write $(2, 7, 1)$ in the basis $F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

3. Write $(1, -1, 0)$ in the basis $F = \left\{ \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

4. Write $(2, 3, 4)$ in the basis $F = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

Find the matrix of the operator with respect to the given basis

1. Give the matrix of $L(x, y, z) = (3x+y+z, 5x-2y+z, y+z)$ with respect to $F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

2. Give the matrix of $L(x, y, z) = (2x+3y-z, 4x-y+3z, 2x+z)$ with respect to $F = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

3. Give the matrix of $L(x, y, z) = (-x+4y+2z, 3x-5y+2, 3x+2y)$ with respect to $F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

4. Give the matrix of $L(x, y, z) = (2x-y, 3x+y+4z, x+2y+z)$ with respect to $F = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Angles and Magnitudes

1. Compute

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 1 \\ 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \\ 7 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}.$$

2. Find the magnitudes and corresponding unit vectors for

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ -1 \\ -3 \end{bmatrix}.$$

3. Find $\text{proj}_{\mathbf{v}} \mathbf{u}$ for

   (a) $\mathbf{u} = (5, 2), \mathbf{v} = (-3, 4)$

   (b) $\mathbf{u} = (2, 1), \mathbf{v} = (7, 1)$

   (c) $\mathbf{u} = (3, 1, 4), \mathbf{v} = (2, 1, 1)$

   (d) $\mathbf{u} = (2, 1, 1), \mathbf{v} = (-4, -1, -1)$

   (e) $\mathbf{u} = (5, 0, 0), \mathbf{v} = (3, 2, 1)$.

Diagonalization Theory

1. In class we saw that

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

   Multiply out the three matrices on the right and confirm that this works.

2. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What are the eigenvalues of A? Is $A^2 = A$? Why not?

3. Show the following pairs of matrices are not similar:

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 5 & 1 & 3 \end{bmatrix}$$

$$E = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 8 & -2 \\ 0 & 0 & 10 \end{bmatrix} \qquad F = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 5 & 0 \\ 5 & 3 & 12 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

Find the characteristic polynomials, eigenvalues (with algebraic multiplicity), and bases for the eigenspaces, of the following matrices.

1. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Determinants

1. Find all values of k for which $A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$ is invertible.

2. Compute the determinants of:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{bmatrix}$$

Diagonalization

For each of the following matrices, determine whether it is diagonal. If it is, diagonalize it, then computeA^5.

1. $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$

2. $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$

3. $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

Orthogonality and Projection

1. Suppose $\|\mathbf{u}\| = 3, \|\mathbf{u}+\mathbf{v}\| = 4, \|\mathbf{u}-\mathbf{v}\| = 6$. Find $\|\mathbf{v}\|$.

2. Find the orthogonal complement (in $\mathbb{R}^n$) of the following spaces:

$$W = \{(2t, -t) : t \in \mathbb{R}\}$$
$$W = \text{span}\{(2, -1, 3)\}$$
$$W = \{(t, -t, 3t) : t \in \mathbb{R}\}$$
$$W = \text{span}\{(1, -1, 3, -2), (0, 1, -2, 1)\}.$$

3. Find the orthogonal decomposition of

   (a) $(7, -4)$ with respect to span$\{(1, 1)\}$

   (b) $(1, 2, 3)$ with respect to span$\{(2, -2, 1), (-1, 1, 4)\}$

   (c) $(4, -2, 3)$ with respect to span$\{(1, 2, 1), (1, -1, 1)\}$

   (d) $(3, 2, -3, 4)$ with respect to span$\{(2, 1, 0, 1), (0, -1, 1, 1)\}$.