Math 214 Test 1 Solutions

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Problem 1.

(a) (10 points) Find the set of solutions to the following system of linear equations:

$$\begin{aligned} 3x + 7y + 5z &= 34 \\ 2x + 4y + 2z &= 20 \\ -x + 3z &= -2 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 3 & 7 & 5 & 34 \\ 2 & 4 & 2 & 20 \\ -1 & 0 & 3 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 14 \\ 2 & 4 & 2 & 20 \\ -1 & 0 & 3 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 14 \\ 0 & -2 & -4 & -8 \\ 0 & 3 & 6 & 12 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the set of solutions is $\{(2 + 3\alpha, 4 - 2\alpha, \alpha)\}$.

(b) (10 points) Find the nullspace of the following matrix: $\begin{bmatrix} -2 & 1 & -2 \\ 4 & -1 & -2 \\ 5 & -2 & 2 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & -1/2 & 1 \\ 4 & -1 & -2 \\ 5 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & 1 \\ 0 & 1 & -6 \\ 0 & 1/2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

so the nullspace is $\{(2\alpha, 6\alpha, \alpha)\}$.

Problem 2.

(a) (10 points) Find the inverse of the matrix $\begin{bmatrix} 2 & -2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 3 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 2 & -2 & 3 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 & -2 \\ 0 & -6 & 1 & 1 & 0 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & -2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & -5 \\ 0 & 1 & 0 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right]$$

so the inverse is

$$\begin{bmatrix} -1 & 4 & -5 \\ 0 & -1/2 & 1 \\ 1 & -3 & 4 \end{bmatrix}.$$

(b) (5 points) If $B^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 6 & 2 & -2 \\ 4 & 1 & 1 \end{bmatrix}$ find the set of solutions to $B\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$.

Solution:

$$\mathbf{x} = B^{-1} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 6 & 2 & -2 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 19 \\ 26 \\ 21 \end{bmatrix}$$

(c) (5 points) Compute

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 4 & 2 \\ -3 & 3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 4 & 2 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 4 \\ 21 & 17 \end{bmatrix}$$

Problem 3 (4 points each)**.** Are the following statements true or false? Give a short (one sentence or less) explanation or counterexample.

(a) Every vector space contains at least one vector.

Solution: True, because every vector space contains the zero vector.

(b) $\{(a, b, c) : a + b = c + 1\}$ is a subspace of $\mathbb{R}^3$.

Solution: False, because the zero vector is not an element.

(c) There is a system of linear equations with exactly two solutions.

Solution: False, because the nullspace has either one or infinitely many elements, and the number of solutions is the number of elements of the subspace.

(d) $\{(f : \mathbb{R} \to \mathbb{R} : f(x) < 5 \text{ for all } x \in \mathbb{R}\}$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Solution: False, because $f(x) = 3$ is in the set, but $2f$ is not in the set since $f(0) = 6$.

(e) A system of equations with more variables than equations always has a solution.

Solution: False. A system like this will *usually* have a solution, but $x + y + z = 0, x + y + z = 1$ has more variables than equations and still has no solution.

Problem 4 (10 points each)**.**

1. Prove that $S = \{(a, b, c) : 2a + 3b - c = 0\}$ is a subspace of $\mathbb{R}^3$.

   Solution: We need to check three things.

   (a) $2 \cdot 0 + 3 \cdot 0 - 0 = 0$ so $\mathbf{0}$ is in the set.

   (b) If $(a, b, c), (d, e, f)$ are elements, then $2a + 3b - c = 0$ and $2d + 3e - f = 0$, so $2(a + d) + 3(b + e) - (c + f) = 0$ and thus $(a, b, c) + (d, e, f)$ is an element.

   (c) If (a, b, c) is an element, then $2a + 3b - c = 0$, so $2(ra) + 3(rb) - rc = 0$, so $r(a, b, c)$ is an element.

   Thus by the subspace theorem, this is a subspace.

2. Prove that $T = \{a_0 + a_1 x + a_2 x^2 : a_0 = a_1\}$ is a subspace of $\mathcal{P}_2(x)$.

   Solution: We need to check three things.

(a) $0 = 0 + 0x + 0x^2$ is an element of the set, since $0 = 0$.

(b) If $a_0 + a_1x + a_2x^2$ and $b_0 + b_1x + b_2x^2$ are elements, then $a_0 = a_1$ and $b_0 = b_1$. Then $a_0 + a_1x + a_2x^2 + b_0 + bx^+ b_2x^2 = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$ is an element since $a_0 + b_0 = a_1 + b_1$.

(c) If $a_0 + a_1x + a_2x^2$ is an element and r is a scalar, then $r(a_0 + a_1x + a_2x) = ra_0 + ra_1x + ra_2x^2$ is an element since $ra_0 = ra_1$.

Thus by the subspace theorem this is a subspace.

Problem 5 (10 points each)**.**

1. Is $(3, 2, 5)$ in the span of the set $S = \{(1, 1, 1), (1, 2, 3), (3, 5, 7)\}$?

   Solution: We set up the system of equations

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 1 & 2 & 5 & 2 \\ 1 & 3 & 7 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

   and since the last equation is $0 = 4$ we have a contradiction. Thus no solution exists to the system of equations, and $(3, 2, 5)$ is not in the span of S.

2. Suppose U, W are subspaces of a vector space V. We define $U \cap W = \{\mathbf{v} : \mathbf{v} \in U \quad \text{and} \quad \mathbf{v} \in W\}$ to be the set of all vectors that are in both U and W, which we call the “intersection” of U and W.

   Prove that $U \cap W$ is a subspace of V.

   Solution: We need to check three things.

   (a) By definition of subspace, we know that $\mathbf{0} \in U$ and $\mathbf{0} \in W$. Thus $\mathbf{0} \in U \cap W$.

   (b) Suppose $\mathbf{v}_1, \mathbf{v}_2 \in U \cap W$. Then $\mathbf{v}_1, \mathbf{v}_2 \in U$, so $\mathbf{v}_1 + \mathbf{v}_2 \in U$ by additive closure. Similarly, $\mathbf{v}_1 \in W, \mathbf{v}_2 \in W$, so $\mathbf{v}_1 + \mathbf{v}_2 \in W$ by additive closure. Thus $\mathbf{v}_1 + \mathbf{v}_2 \in U \cap W$ by definition.

   (c) Suppose $\mathbf{v} \in U \cap W$ and $r \in \mathbb{R}$. Then $\mathbf{v} \in U$ so $r\mathbf{v} \in U$ by scalar multiplicative closure; and $\mathbf{v} \in W$ so $r\mathbf{v} \in W$ by scalar multiplicative closure. Thus $r\mathbf{v} \in U \cap W$ by definition.

   Thus by the subspace theorem, $U \cap W$ is a subspace of V.