Math 214 Test 2 Solutions

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Problem 1.

(a) Find a basis for $\mathbb{R}^3$ containing $\begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, and prove it is a basis.

Solution: These vectors are linearly independent, so we just need to find a third vector that isn't in the span. It looks like $(0, 0, 1)$ should be linearly independent, and indeed we can row-reduce

$$\begin{bmatrix} 3 & 6 & -1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to see these three vectors are linearly independent. Since we have a linearly independent set of three vectors in a three-dimensional space, it is a basis. Thus $\{(3, 6, -1), (2, 1, 0), (0, 0, 1)\}$ is a basis for $\mathbb{R}^3$.

(b) Find a basis for $\mathbb{R}^3$ that is a subset of

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \right\}$$

and prove it is a basis.

Solution: We need to find out which vector depends on the others. We may notice that

$$\begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix},$$

in which case we know we can abandon the last vector. If we don't see that we can row-reduce:

$$\begin{bmatrix} 1 & 5 & -1 & 6 \\ 2 & 2 & -3 & 5 \\ 3 & -1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & 6 \\ 0 & -8 & -1 & -7 \\ 0 & -16 & 1 & -17 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & 6 \\ 0 & 1 & 1/8 & 7/8 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

and we already see that the first three columns will contain leading ones. Thus either way, we see that

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix} \right\}$$

is still a spanning set for $\mathbb{R}^3$. Since it has three elements and $\dim \mathbb{R}^3 = 3$, it is also a basis.

Problem 2.

Let $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 5x + 2y - z \\ 3x + -3z \\ 2y + 4z \end{bmatrix}.$

(a) Prove that L is a linear transformation of L.

(b) Find a matrix for L.

(c) Find a basis for $\ker(L)$.

(d) Find a basis for the image $L(\mathbb{R}^3)$.

(e) If L is invertible, find a (non-matrix!) formula for the inverse.

Solution:

1.

2. $\begin{bmatrix} 5 & 2 & -1 \\ 3 & 0 & -3 \\ 0 & 2 & 4 \end{bmatrix}$

3.

$$\begin{bmatrix} 5 & 2 & -1 \\ 3 & 0 & -3 \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -6 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so the kernel has basis $\{(1, -2, 1)\}$.

4. The image has basis $\{(5, 3, 0), (2, 0, 2)\}$.

5. The operator isn't invertible becasue it has nontrivial nullspace.

Problem 3.

Let $T : \mathcal{P}_2(x) \to \mathbb{R}^3$ be given by $T(f) = \begin{bmatrix} f(0) \\ f(2) - f(0) \\ 2f(-2) \end{bmatrix}$.

(a) Prove T is a linear transformation.

(b) Find a matrix for T with respect to the standard bases $\{1, x, x^2\}$ and $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for each space.

(c) Find a basis for $\ker(T)$.

(d) Find a basis for the image $T(\mathcal{P}_2(x))$.

(e) If T is invertible, find a (non-matrix!) formula for the inverse.

Solution:

1.

2. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 2 & -4 & 8 \end{bmatrix}$

3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 2 & -4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So the kernel is trivial and has basis $\{\}$.

4. The map is onto. We can look at the three columns of the original matrix and say that we have basis $\{(1, 0, 2), (0, 2, -4), (0, 4, 8)\}$; but we could also note that the image is all of $\mathbb{R}^3$ and take the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

   To get a basis for the actual image we need to give either $\{1 + x^2, 2x - 4x^2, 4x + 8x^2\}$ or $\{1, x, x^2\}$

5. To invert the matrix we can row-reduce again. We get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 2 & -4 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & -1 & 2 & -1/2 & 0 & 1/4 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & 4 & -1/2 & 1/2 & 1/4 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/4 & 1/4 & -1/8 \\ 0 & 0 & 1 & -1/8 & 1/8 & 1/16 \end{array}\right]$$

so the inverse matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1/4 & -1/8 \\ -1/8 & 1/8 & 1/16 \end{bmatrix}$. This means the inverse function is

$$T^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a + \frac{1}{8}(2a + 2b - c)x + \frac{1}{16}(-2a + 2b + c)x^2.$$

Problem 4.

(a) Find bases for the rowspace, columnspace, and nullspace of

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ -1 & 2 & 1 & 3 \\ 2 & 14 & 7 & 12 \end{bmatrix}$$

Solution: We have

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ -1 & 2 & 1 & 3 \\ 2 & 14 & 7 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 6 & 3 & 6 \\ 0 & 6 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the rowspace has basis $\{(1, 0, 0, -1), (0, 2, 1, 2)\}$. The columnspace has basis $\{(1, -1, 2), (4, 2, 14)\}$. And the kernel has basis $\{(1, -1, 0, 1), (0, -1, 2, 0)\}$.

rowspace $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \right\}$ columnspace $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} \right\}$ nullspace $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

(b) Let U be a finite-dimensional vector space, let $L : U \to U$ be a linear transformation, and suppose that $L(\mathbf{u}) = \mathbf{0}$ for some $\mathbf{u} \neq \mathbf{0}$. Prove that L is not onto (surjective).

Solution: We know that $\mathbf{u} \in \ker(L)$, so $\dim \ker(L) > 0$. But by the Rank-Nullity Theorem, we know that $\dim \ker(L) + \dim L(U) = \dim(U)$, so we have $\dim(U) > \dim L(U)$. Thus $L(U) \neq U$, so L is not onto.

Problem 5.

(a) If $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, prove that $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

Solution: Suppose

$$\mathbf{0} = a(\mathbf{u} + \mathbf{v}) + b(\mathbf{u} - \mathbf{v}) = (a + b)\mathbf{u} + (a - b)\mathbf{v}.$$

Then since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, we know that $a + b = 0$ and $a - b = 0$. Solving this gives us $a = b$, and thus $2a = 0$ so $a = 0$, and then $b = 0$. So whenever a linear combination of $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is equal to zero, the coefficients must be zero; thus the set is linearly independent by definition.

(b) Suppose $S : U \to V$ and $T : V \to W$ are both injective. Define $L : U \to W$ by $L(\mathbf{u}) = T(S(\mathbf{u}))$. Prove that L is injective.

Solution: Suppose $L(\mathbf{u}) = L(\mathbf{v})$. Then $T(S(\mathbf{u})) = T(S(\mathbf{v}))$. Since T is injective, we know that $S(\mathbf{u}) = S(\mathbf{v})$, and then since S is injective we know that $\mathbf{u} = \mathbf{v}$. Thus whenever $L(\mathbf{u}) = L(\mathbf{v})$, we know that $\mathbf{u} = \mathbf{v}$, and thus by definition L is injective.