MATHEMATICS COMPREHENSIVE EXAMINATION, SPRING 2005

NAME: _____ Start Time: _____

Each question is worth 4 points. You have three hours.

Single Variable Calculus

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1. The initial value problem $A'(t) = t^2 + A$, $A(2) = 3$ defines some function $A(t)$. Find an equation for the tangent line to $A(t)$ at $t = 2$, and use it to approximate $A(2.3)$.

2. A swimming pool currently contains 5 gallons of bacteria. The filtration system removes bacteria at a rate of 2 gallons a day, but the bacteria are constantly reproducing at a rate of 30% per day. Write down an initial value problem modeling the amount B of bacteria (in gallons) in the pool after t days (see question 1 above for an example of an initial value problem). [Note that you are not asked to find the function $B(t)$ explictly.]

3. If $f(x) = e^{2x} \sin(\ln x^2)$, find $f'(x)$.

4. The material for the sides of a rectangular picture frame costs 5 cents per inch; the material for the top and bottom of the frame costs 10 cents per inch. Find the dimensions of the least expensive frame that will enclose a picture with area 200 in^2. Include an argument that you have really found the *minimal* cost.

5. (a) Write down the definition of $f'(a)$:

$$f'(a) =$$

(b) What feature of the graph below does $f'(a)$ measure?

(c) Explain why your answer in (a) is a reasonable definition for the feature in (b). This should require no more than two or three sentences, or 20-30 words.

6. Express the area of the region enclosed by the graphs of $y = x^3 - 3x^2$, $y = 6 - 2x$, $x = 0$, and $x = 3$ as an integral. You do not need to evaluate the integral.

7. Consider the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$.

   (a) Can the Ratio Test be used to determine that this series converges? Justify your answer by explaining what happens when you apply the Ratio Test to this series in 15-20 words.

   (b) Use the Integral Test to show that this series converges. Make sure you briefly explain why your calculation(s) show the series converges.

8. Given the following information about an unknown function $g(x)$:

$$\int_1^2 \frac{g(u)}{u}\, du = 3, \quad \int_1^2 g(u)\, du = 4, \quad \int_1^4 g(u)\, du = 5, \quad g(1) = 2, \quad g(2) = -2,$$

   (a) Evaluate $I = \displaystyle\int_1^2 \ln(x) g'(x)\, dx$.

   (b) Evaluate $J = \displaystyle\int_1^2 x g(x^2)\, dx$.

9. The following table gives the measurements (in inches) of the width of an ink blot at one-half inch intervals.

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	4	7	2	0	2.5	4.5	5

(a) Which of the following numerical integration techniques would you expect to produce the most accurate approximation of the area of the ink blot, $\int_0^3 f(x)\ dx$: left Riemann sum, right Riemann sum, Trapezoidal Rule or Simpson's Rule? Explain your answer in 15-20 words.

(b) Now, use the right Riemann sum approximation with $n = 6$ steps to estimate $\int_0^3 f(x)\ dx$.

10. Evaluate the definite integral

$$\int_0^{\pi} \frac{\sin x}{2 + \cos x}\ dx$$

11. Find:

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\, dy\, dx$$

12. Say you're walking along the surface $f(x,y) = 4 - x^2 - 2y^2$. You happen to be at the point $(1,-1,1)$ right now.

    (a) What specific direction should you travel from this point to climb up the surface the fastest (steepest ascent)? Your answer should be a 2-dimensional vector.

    (b) At what rate would you be climbing up/down the surface if you headed in the $\vec{v} = 3\vec{i} + 4\vec{j}$ direction from this point?

13. Given the contour diagram for $f(x, y)$ shown, answer the following.

    (a) Estimate $f_x(4, 1)$. Clearly show your work as to how you are making this estimate.

    (b) Fill in the blanks with $>$, $<$, or $=$.

    $f_{xx}(4, 1)$ _____ 0

    $f_{xy}(4, 1)$ _____ 0

14. Explain your answer to the following: Do you expect the line integral of the pictured vector field over the given directed curve to be positive, negative, or zero? [Your clear explanation of 15-20 words explaining why is worth more than the answer to the question.]

15. Compute the line integral of the vector field $\vec{F}(x, y) = (6xy + y^2, 3x^2 + 2xy)$ over the line segment from $(1, 2)$ to $(-1, -2)$.

16. Give an example (without proof) of a pair of subspaces V and W of $\mathbf{R}^3$ such that $V \neq \{\vec{0}\}$, $V \neq W$, $V \subset W$, and $W \neq \mathbf{R}^3$.

17. (a) Give the definition of what it means for a vector $\vec{v}$ to be an eigenvector of a matrix A.

(b) Prove or disprove the following statement: If $\vec{v}$ is an eigenvector of a matrix A, then $2\vec{v}$ is also an eigenvector of A.

18. Let $\vec{v}$ be a nonzero vector in $\mathbf{R}^3$, and let P be a plane in $\mathbf{R}^3$ containing the origin, such that $\vec{v} \notin P$ and $\vec{v}$ is not orthogonal to P. Let $\vec{w} = \text{proj}_P(\vec{v})$, the projection of $\vec{v}$ onto P. Answer the following without need for proof or explanation:

(a) The vector $\vec{w}$ is (circle all that apply):

(i) in P (ii) perpendicular to P (iii) perpendicular to $\vec{v}$ (iv) parallel to $\vec{v}$

(b) The vector $\vec{v} - \vec{w}$ is (circle all that apply):

(i) in P (ii) perpendicular to P (iii) perpendicular to $\vec{v}$ (iv) parallel to $\vec{v}$

(c) If $\vec{v} = [1\ 2\ 3]$ and P is the xz-plane, what is $\vec{w}$?

19. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$. Let $B = \text{rref}(A)$, the reduced row echelon form of A.

(a) Find B.

(b) Do A and B have the same nullspace? Why or why not?

(c) Do the systems of equations $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ have the same solutions? Why or why not?

20. (a) Prove or disprove: For any pair of $n \times n$ matrices A and B, $\det(A + B) = \det(A) + \det(B)$.

(b) Prove or disprove: If A and B are $n \times n$ matrices and $\det(A) = 0$, then AB is not invertible.

21. Let A be a set with n elements. How many elements are in the power set $P(A)$? Prove your result using induction.

22. Let A be a set, and let $\sim$ be a relation on A. Define specifically what it means for $\sim$ to be an equivalence relation on A [don't just use the relevant terms, explain what the terms mean].

    For $a, b \in \mathbf{R}$, define $a \sim b$ if and only if $|a - b| \leq 1$. Prove or disprove that $\sim$ defines an equivalence relation on the set of reals $\mathbf{R}$.

23. Let A and B be finite sets with $|A| < |B|$. Decide whether these statements are True or False and give a reason (at least a sentence) to defend your answer.

    a) There is a one-to-one mapping of A into B.

    b) There is a one-to-one map of A onto B.

    c) There is a one-to-one map of B into A.

    d) There is a function mapping B onto A.

24. Prove that the four-digit number $n = abcd$ is divisible by 11 if and only if $a - b + c - d$ is divisible by 11.

25. A certain class consists of 12 men and 16 women. How many committees can be chosen from this class consisting of each of the following. [Note: You may leave your answers in combination or permutation form, or in terms of factorial notation.]

    (a) 7 people

    (b) 3 men and 4 women

    (c) 7 women or 7 men