Linear Algebra (*Math 214*) Common Topics List[1]

1. **Computations and Proofs**
   (a) The course (lectures, homework, exams) should include computations and proofs
2. **Geometry and Algebra of Vectors**
   (a) vector addition
   (b) scalar multiplication
   (c) linear combinations
3. **Dot Product**
   (a) length (norm) of vectors
   (b) unit vectors and normalizing
   (c) angle between vectors
   (d) projecting onto a vector
   (e) Cauchy-Schwarz Inequality, i.e. $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}|| \; ||\vec{y}||$
   (f) Triangle Inequality
   (g) orthogonal vectors
4. **Lines and Planes**
   (a) vector form, general form of equation of line; parametric equations
   (b) vector form, normal form, general form of equation of a plane
5. **Solving Systems of Linear Equations**
   (a) linear equations vs. nonlinear equations
   (b) coefficient matrix, augmented matrix
   (c) row echelon form, reduced row echelon form
   (d) Gaussian Elimination, Gauss-Jordan Elimination
   (e) leading variables, free variables
   (f) elementary row operations, row equivalent matrices
   (g) homogeneous system vs. nonhomogeneous
6. **Spanning Sets and Linear Independence**
   (a) the span of a set of vectors
   (b) linearly dependent, linearly independent

[1] This list was updated at the 10/24/11 Math Departmental retreat

7. **Matrix Operations**
    (a) identity matrix, zero matrix
    (b) matrix sum, matrix product, scalar multiplication
    (c) transpose of a matrix
    (d) symmetric matrices
8. **Matrix Algebra**
    (a) matrix multiplication is noncommutative
    (b) associativity, left and right distributivity, multiplicative identity
    (c) properties of the transpose (e.g., $(AB)^T = B^T A^T$)
9. **Inverse of a Matrix**
    (a) definition of inverse and invertible
    (b) properties of invertible matrices (e.g., $(AB)^{-1} = B^{-1}A^{-1}$)
    (c) elementary matrices
    (d) Equivalent conditions for when a matrix is invertible (non-singular)
10. **Subspaces, Basis, Dimension, and Rank**
    (a) definition of subspace (in $\mathbb{R}^n$)
    (b) subspace spanned by a set of vectors
    (c) row space of A, column space of A, null space of A
    (d) rank of A, nullity of A
    (e) basis of a subspace
    (f) dimension of a subspace
    (g) standard basis of $\mathbb{R}^n$
    (h) finding a basis for row space, column space, and null space of a matrix
    (i) For $A_{m\times n}$, $\mathsf{rank}(A) + \mathsf{nullity}(A) = n$
    (j) $A\vec{x} = \vec{b}$ has a solution iff $\vec{b}$ is in column space of A.
11. **Determinants**
    (a) computing determinants (expanding along rows or columns; cofactors)
    (b) how elementary row operations affect the determinant
    (c) A is singular $\iff \det(A) = 0$
    (d) theorems about determinants (e.g., $\det(AB) = \det(A)\det(B)$)
12. **Eigenvalues and Eigenvectors**

(a) definitions of eigenvalues and eigenvectors

(b) characteristic polynomial & characteristic equation

(c) the eigenvalues are the roots of the characteristic polynomial

(d) theorems about eigenvalues, like:

   i. {eigenvalues of A } = {eigenvalues of A^T } (b/c same char. poly.)

   ii. If A is nonsingular (so that A^{-1} exists), then
   λ is an eigenvalue of $A \iff \lambda^{-1}$ is an eigenvalue of A^{-1} (use same eigenvector)

   iii. λ is an eigenvalue of $A \iff \lambda - c$ is an eigenvalue of $A - cI$ (use same eigenvector)

(e) A is singular $\iff$ 0 is an eigenvalue of A

(f) Diagonalizable (definition)

(g) If A has *distinct* eigenvalues, then A is diagonalizable.

(h) $\det(A) = \prod \lambda_i$ (the determinant is the product of the eigenvalues)

13. **Similarity**

(a) similar matrices

(b) if A and B are similar, then they have the same characteristic polynomial, and therefore the same eigenvalues (& other such theorems)

14. **Projections onto Subspaces and Orthogonal Complements**

(a) orthogonal complement of a subspace

(b) Properties of $W^\perp$ like:

   i. if W is a subspace, $W^\perp$ is also a subspace

   ii. $(W^\perp)^\perp = W$

(c) $\mathsf{null}(A)$ is the orthogonal complement of $\mathsf{row}(A)$, i.e., $(\mathsf{row}(A))^\perp = \mathsf{null}(A))$

(d) given a vector $\vec{b}$ and a subspace W, find $\vec{b}_W$ and $\vec{b}_{W^\perp}$ (where $\vec{b}_W$ is the projection of $\vec{b}$ onto W)

   i. be aware that $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$ and that this "decomposition" is *unique*

15. **Orthogonal Bases**

(a) orthogonal set of vectors (orthogonal basis)

(b) orthonormal basis

(c) Gram-Schmidt process

Optional Topics for Linear Systems *Math 214*

Orthogonal Matrices and Orthogonal Diagonalization of Symmetric Real Matrices

- definition: $A^T A = I$
- properties of orthogonal matrices (e.g., the columns form an *orthonormal* basis for $\mathbb{R}^n$ (if $A_{n\times n}$), etc.)
- If A is orthogonal then $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$, etc.
- finding an orthogonal diagonalization of a real symmetric matrix
- Spectral theorem (about symmetric matrices)

Abstract Vector Spaces and Linear Transformations

LU **Factorization**

QR **Factorization**

Least Squares Approximation

Matrix Block Multiplication

Cramer's Rule

Gram-Schmidt Process

Markov Chains

Iterative Methods

Singular Value Decomposition

Cayley-Hamilton Theorem