NOTES

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On Cantor's First Uncountability Proof, Pick's Theorem, and the Irrationality of the Golden Ratio

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Abstract[redacted]

In 1874, two years before the publication of his famous diagonalization argument, Georg Cantor's first proof of the uncountability of the real numbers appeared in print [**1**]. Surprisingly, a small twist on Cantor's line of reasoning shows that the golden ratio is irrational, as we shall demonstrate herewith. En route, we will make use of another classic 19th century theorem by another Georg, this time Georg Pick. Pick's theorem [**2**] provides a simple formula for the number of lattice points enclosed within a simply connected polygonal region in the plane with lattice point vertices.

We begin by recapitulating Cantor's 1874 proof. To show that the real numbers are uncountable, we must show that given any countable sequence of distinct real numbers, there exists another real number not in the sequence. Like the diagonalization argument, our proof will do so by providing an explicit alogrithm which produces such a number; unlike the diagonalization argument, our proof will employ not decimal expansions but order properties of the real numbers.

Let (a_n) be a countable sequence of distinct real numbers. Suppose that there are two distinct terms a_j and a_k such that no term a_ℓ lies strictly between a_j and a_k—in other words, suppose that $\{a_n\}$ does not possess the intermediate value property. Let L be any real number strictly between a_j and a_k, for example $(a_j + a_k)/2$. Then L is not in the sequence (a_n).

Now suppose that (a_n) does have the intermediate value property. Cantor recursively constructs two new sequences (b_n) and (c_n) as follows. Let $b_1 = a_1$, and let $c_1 = a_2$. Let b_{k+1} be the first term in (a_n) that lies strictly between b_k and c_k. Let c_{k+1} be the first term in (a_n) that lies strictly between b_{k+1} and c_k.

We consider only the case $a_1 > a_2$; the proof in the case $a_2 > a_1$ is very similar. Since $a_1 > a_2$, we get that (b_n) is a strictly decreasing sequence, while (c_n) is strictly increasing. Moreover, every c_n is less than every b_m. Furthermore, note that if $b_n = a_k$ and $b_{n+1} = a_\ell$, then $k < \ell$; a similar statement holds for the sequence (c_n). In other words, as we go along choosing b's and c's, we select them from deeper and deeper in

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the sequence (a_n). Let L be the least upper bound of $\{c_n\}$. Note that $c_k < L < b_\ell$ for all k, ℓ.

We claim that L is not in the sequence (a_n). Suppose otherwise. Then $L = a_\ell$ for some ℓ. Choose m such that $b_m = a_k$ and $c_m = a_r$ with $k, r > \ell$; this is possible because the b's and c's are always coming from deeper and deeper in the sequence (a_n). By construction of the b and c sequences, for every $i \le \max\{k, r\}$, we have $a_i \le c_m$ or $a_i \ge b_m$. But from above, $c_m < L < b_m$. We have thus arrived at a contradiction and hence the conclusion of a rather ingenious proof.

Let's now run through Cantor's argument not with an arbitrary sequence (a_n) but with a very specific sequence. Namely, let (a_n) be the standard enumeration of the set of rational numbers that are greater than 0 and less than or equal to one. That is, the sequence (a_n) is obtained by writing all these rationals in lowest terms, then listing them in order of increasing denominators, where fractions with the same denominator are listed in order of increasing numerators. The first several terms of (a_n) are $1/1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, 5/6, \ldots$

Taking (b_n) and (c_n) as above, a straightforward calculation shows that the first few terms of (b_n) and (c_n) are:

$$\begin{array}{ll} b_1 = \frac{1}{1} & c_1 = \frac{1}{2} \\ b_2 = \frac{2}{3} & c_2 = \frac{3}{5} \\ b_3 = \frac{5}{8} & c_3 = \frac{8}{13} \\ b_4 = \frac{13}{21} & c_4 = \frac{21}{34} \\ b_5 = \frac{34}{55} & c_5 = \frac{55}{89} \\ \vdots & \vdots \end{array}$$

A surprising pattern has revealed itself—suddenly and without warning, our old friends the Fibonaccis have dropped by for a visit! Our next lemma shows that this pattern holds for all n.

Recall that the Fibonacci sequence (F_n) is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$.

Lemma 1. *For all n, we have that $b_n = F_{2n-1}/F_{2n}$ and $c_n = F_{2n}/F_{2n+1}$.*

Proof. We proceed by induction on n. The base case $b_1 = F_1/F_2$ and $c_1 = F_2/F_3$ is immediate. Now assume $b_k = F_{2k-1}/F_{2k}$ and $c_k = F_{2k}/F_{2k+1}$. We will show that $b_{k+1} = F_{2k+1}/F_{2k+2}$; the proof that $c_{k+1} = F_{2k+2}/F_{2k+3}$ is similar. We must show two things about F_{2k+1}/F_{2k+2}, namely that it lies strictly between b_k and c_k, and that it is the first such term in the sequence (a_n).

The fact that $c_k < F_{2k+1}/F_{2k+2} < b_k$ admits a lovely proof without words. Consider Figure 1. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the vectors in $\mathbb{R}^2$ extending from the origin to (F_{2k}, F_{2k-1}) and (F_{2k+1}, F_{2k}), respectively. The slope of $\mathbf{v}_1 + \mathbf{v}_2$ lies strictly between the slopes of $\mathbf{v}_1$ and $\mathbf{v}_2$, as we see readily from the picture. But the slope of $\mathbf{v}_1$ is b_k, the slope of $\mathbf{v}_2$ is c_k, and the slope of $\mathbf{v}_1 + \mathbf{v}_2$ is $(F_{2k-1} + F_{2k})/(F_{2k} + F_{2k+1}) = F_{2k+1}/F_{2k+2}$.

We now show that F_{2k+1}/F_{2k+2} is the *first* term of (a_n) which lies between b_k and c_k. Again, we realize these ratios as slopes of vectors in the plane—see Figure 2.

Let

$$T = \left\{(x, y) \mid F_{2k+1} \le x \le F_{2k+2} \text{ and } \frac{F_{2k}}{F_{2k+1}} < \frac{y}{x} < \frac{F_{2k-1}}{F_{2k}}\right\}.$$

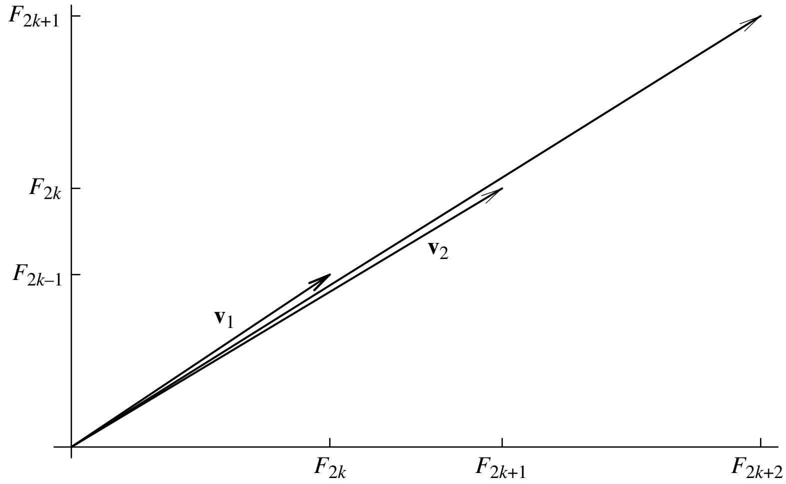

Figure 1. Ratios of consecutive Fibonacci numbers.

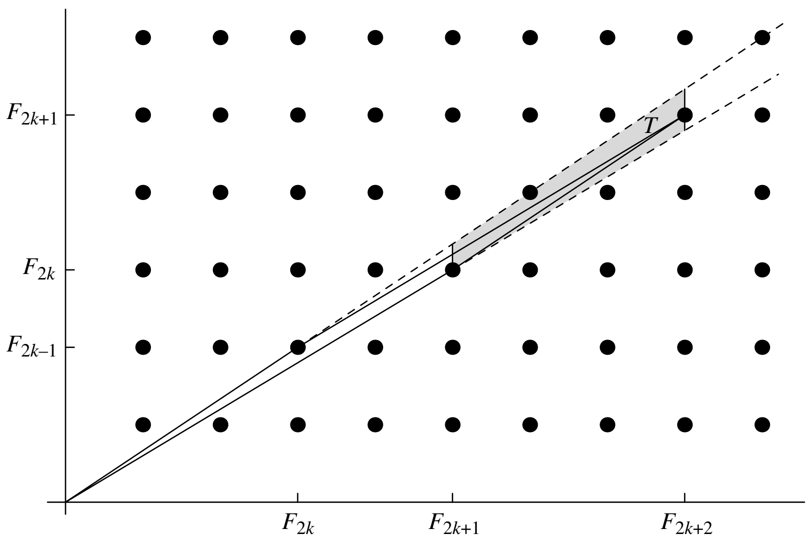

Figure 2. The region T.

The boundary of the shaded region T is a trapezoid. Points on the two vertical line segments (which lie in T) represent fractions with denominators F_{2k+1} and F_{2k+2}. Points on the two dotted lines (which lie outside T) represent the ratios F_{2k-1}/F_{2k} and F_{2k}/F_{2k+1}. Lattice points in Figure 2 represent rational numbers. We claim that T contains no lattice points other than (F_{2k+2}, F_{2k+1}). Any term of the sequence (a_n) which lies between c_k and b_k in magnitude, but comes after b_k and c_k and before F_{2k+1}/F_{2k+2} in the sequence, would be represented by just such a lattice point, and so proving this claim will suffice to complete the proof of Lemma 1. To do so, we invoke the following theorem.

Theorem 2 (Pick's theorem). *Let R be a simply connected polygonal region in $\mathbb{R}^2$ with lattice point vertices. Let A be the area of R; let b be the number of lattice points on the boundary of R; and let i be the number of lattice points in the interior of R.*

Then

$$A = i + \frac{b}{2} - 1.$$

Note that we cannot apply Pick's theorem directly in our case, as the vertices of T might not be lattice points. We can, however, cover T with the four parallelograms P_1, P_2, P_3, and P_4, as shown in Figure 3, where P_1 is the parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$, and the others are translates of P_1, namely $P_2 = P_1 + \mathbf{v}_2$, $P_3 = P_1 + \mathbf{v}_1$, and $P_4 = P_1 + 2\mathbf{v}_1$. An easy induction shows that $2F_{2k+1} > F_{2k+2}$ and $3F_{2k} > F_{2k+2}$, so indeed $T \subseteq P_1 \cup P_2 \cup P_3 \cup P_4 = P$, as depicted in Figure 3.

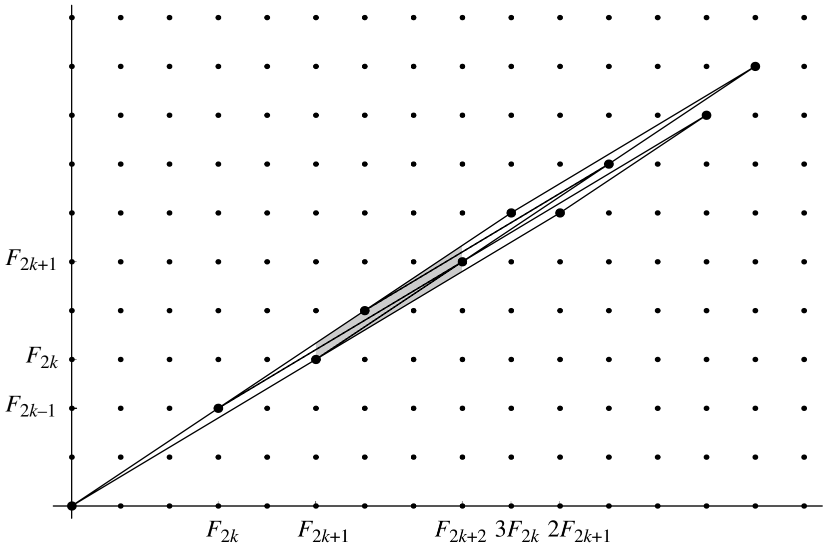

Figure 3. Four parallelograms cover T.

The area of P_1 is

$$\left|\det\begin{pmatrix} F_{2k-1} & F_{2k} \\ F_{2k} & F_{2k+1}\end{pmatrix}\right| = |F_{2k-1}F_{2k+1} - F_{2k}^2| = 1,$$

where the last equality follows from a standard exercise in mathematical induction, namely the fact that $|F_{n-1}F_{n+1} - F_n^2| = 1$ for all integers $n \geq 2$. Since P_1, P_2, P_3, and P_4 are all congruent and intersect only along edges, it follows that P has area 4. Consecutive Fibonacci numbers are coprime—again, an easy induction proves this. It follows that for $j \in \{1, 2, 3, 4\}$, the only lattice points on the boundary of P_j are the vertices of P_j. Hence the boundary lattice points of P are precisely the ten points shown in Figure 3.

So by Pick's theorem, the interior of P contains $4 - 10/2 + 1 = 0$ lattice points. Therefore T contains no lattice points other than (F_{2k+2}, F_{2k+1}), as desired. ■

Let L be the least upper bound of the sequence (c_k), as above. It follows from Lemma 1 that $L = \lim_{k\to\infty} F_{2k}/F_{2k+1}$. Let $\phi = (1 + \sqrt{5})/2$. The number ϕ is called the *golden ratio*. It is well known that the limit L of the ratio of consecutive Fibonacci numbers is ϕ^{-1}. (Quick proof for the uninitiated: Let $M = \lim_{n\to\infty} F_{n+1}/F_n =$

$\lim_{n\to\infty} F_{n+2}/F_{n+1} = \lim_{n\to\infty} (F_{n+1} + F_n)/F_{n+1} = 1 + 1/M$. Solve for M to get $M = \phi$, and then take reciprocals.)

Cantor's line of reasoning showed that L is not an element of $\{a_n\}$. But $\{a_n\}$ contains every rational number between 0 and 1. As $0 < \phi^{-1} < 1$, we therefore conclude that ϕ^{-1} is not rational. Hence we have the following theorem.

Theorem 3. *The golden ratio is irrational.*

We remark that our discussion will immediately remind many readers of the continued fraction expansion for ϕ. Indeed, our proof that the two sequences produced by Cantor's method are given by ratios of consecutive Fibonacci numbers tracks closely along the lines of a proof that a truncated continued fraction gives a best approximation amongst all rationals with equal or smaller denominator.

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The Diophantine Equation $x^4 \pm y^4 = iz^2$ in Gaussian Integers

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Abstract. In this note we find all the solutions of the Diophantine equation $x^4 \pm y^4 = iz^2$ using elliptic curves over $\mathbb{Q}(i)$. Also, using the same method we give a new proof of Hilbert's result that the equation $x^4 \pm y^4 = z^2$ has only trivial solutions in Gaussian integers.

1. INTRODUCTION. The Diophantine equation $x^4 \pm y^4 = z^2$, where x, y, and z are integers, was studied by Fermat, who proved that there exist no nontrivial solutions. Fermat proved this using the *infinite descent* method, proving that if a solution can be found, then there exists a smaller solution (see for example [**1**, Proposition 6.5.3]).

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