- Tangent planes and linear approximations
- Gradients and directional derivatives: $f_{\vec{u}} = \nabla f \cdot \vec{u}$.
- Contour maps and first and second derivatives
- Max and min

  Critical points where gradient is zero—intuition.

  Local extrema and saddle points. $x^2 + y^2, x^2 - y^2$.

  Second partial test: Secretly eigenvalues! $D = f_{xx} f_{yy} - f_{xy}^2$ is determinant of matrix. If $D > 0$ and $f_{xx} > 0$, min. If $D > 0$ and $f_{xx} < 0$,then max. If $D < 0$ then saddle. If $D = 0$, know nothing.

  Global extrema and EVT.

  Lagrange: To maximize f subject to $g = c$, solve $\nabla f = \lambda \nabla g$ and $g = c$. (Guarantees that gradient is perpendicular to $g = c$ surface, which means that directional derivative is zero). λ tells us how the optimum changes with c.

  Let $f(x, y) = x + y$ with $x^2 + y^2 = 1$. Then our equation is $\vec{i} + \vec{j} = 2x\lambda\vec{i} + 2y\lambda\vec{j}$ and thus $x = y = 1/(2\lambda)$. Then we have $2x^2 = 1$ so $x = y = \pm\sqrt{2}/2$ gives us our local extrema.

- Double integrals and limit swapping

  What order can you write limits in? Variables only on inside.

  Can usually switch. (exceptions require something to be unbounded and don't really come up).

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3 + 1}\, dy\, dx.$$

  The integral with respect to y is a huge pain, so we don't do it. We sketch the region: x goes from 0 to 6, and y goes from $x/3$ to 2. We can turn this around to say: y goes from 0 to 2, and x goes from 0 to $3y$. So we get

$$\begin{aligned} \int_0^2 \int_0^{3y} x\sqrt{y^3 + 1}\, dx\, dy &= \int_0^2 \left(x^2/2\sqrt{y^3 + 1}\Big|_0^{3y} \right) dy \\ &= \int_0^2 \left(9y^2/2\sqrt{y^3 + 1} \right) dy \\ &= (y^3 + 1)^{3/2}\Big|_0^2 = 27 - 1 = 26. \end{aligned}$$

Bonus problem:

$$\int_0^2 \int_y^2 e^{x^2}\,dx\,dy = \int_0^2 \int_0^x e^{x^2}\,dy\,dx$$
$$= \int_0^2 xe^{x^2}\,dx = e^{x^2}/2|_0^2 = e^4/2 - 1/2.$$

- Line integrals and conservative vector fields

  Vector field: $\mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbb{R}^3 \to \mathbb{R}^3$. Puts a vector at each point. (Force field, gravity field, direction to your destination).

  Line integral: computes work moving through vector field.

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt.$$

Doesn't depend on the parametrization as long as you go over the same path.

Let $F(x,y) = x\vec{i} + y\vec{j}$ and let our curve C be the straight line from $(1,0)$ to $(0,2)$. We can parametrize this with $\vec{r}(t) = (1-t)\vec{i} + 2t\vec{j}$ for $0 \le t \le 1$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$
$$= \int_0^1 \Big((1-t)\vec{i} + (2t)\vec{j}\Big) \cdot (-\vec{i} + 2\vec{j})\,dt$$
$$= \int_0^1 (t-1) + 4t\,dt = \int_0^1 5t - 1\,dt$$
$$= 5t^2/2 - t|_0^1 = 3/2.$$

Now let C_2 be the parabola with vertex at $(0,2)$ that goes through $(1,0)$, going counterclockwise. The equation for this parabola is $y = 2 - 2x^2$, which we can parametrize by $\vec{r}_2(t) = t\vec{i} + (2-2t^2)\vec{j}$ as t goes from 1 to 0. Then we have

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_1^0 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$
$$= \int_1^0 \Big(t\vec{i} + (2-2t^2)\vec{j}\Big) \cdot (\vec{i} - 4t\vec{j})\,dt$$
$$= \int_1^0 t - 8t + 8t^3\,dt = \int_1^0 8t^3 - 7t\,dt$$
$$= 2t^4 - 7t^2/2|_1^0 = 7/2 - 2 = 3/2.$$

Notice that this is a different integral—over a different path—but we get the same result!

This is because $\vec{F}$ is a conservative vector field. If F is a conservative vector field then it has path independence. F is conservative if and only if $F = \nabla f$ for some $F : \mathbb{R}^n \to \mathbb{R}$. We say f is a potential function. In this case,

$$\int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P).$$

We can see if a field is conservative from a picture. Do closed curves have zero integral?

We can also test it with the curl: it's always true that $\nabla \times \nabla f = \vec{0}$, so F is conservative if and only if $\nabla \times F = \vec{0}$.

You can find the potential function in two ways: One is to define

$$f(Q) = \int_{C_Q} \vec{F} \cdot d\vec{r}(t)\, dt +$$

for C_Q any path starting at the origin and ending at Q.

The other is basically by solving differential equations. Let $\vec{F}(x, y) = (2xy - 1)\vec{\imath} + x^2\vec{\jmath}$. Suppose F is conservative. Then there's a potential function $f(x, y)$ such that $\frac{\partial f}{\partial x} = 2xy - 1$ and $\frac{\partial f}{\partial y} = x^2$. By integrating each equation, we get

$$f(x, y) = x^2y - x + g(y) \qquad\qquad f(x, y) = x^2y + h(x).$$

This tells us that $f(x, y) = x^2y - x + C$.