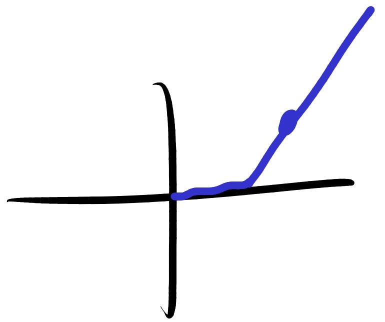
Related Rates

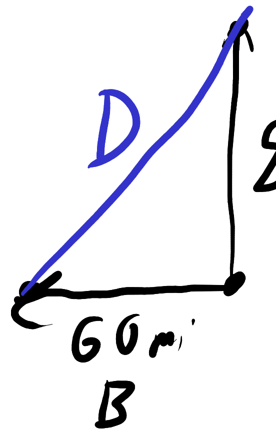
f' as rate of change

implicit functions



Example 2.46. Suppose one car drives north at 40 mph, and an hour later another starts driving west from the same place at 60 mph. After a second hour, how quicly is the distance between them increasing?

Q: what's $D'(2)$



$A(2) = 80\ mi$ $\quad A'(2) = 40\ mi/h$

$B(2) = 60\ mi$ $\quad B'(2) = 60\ mi/h$

$D(t)^2 = A(t)^2 + B(t)^2$

$D(2)^2 = A(2)^2 + B(2)^2 = 80^2 mi^2 + 60^2 mi^2 = 10{,}000\ mi^2$

$\Rightarrow D(2) = 100\ mi$

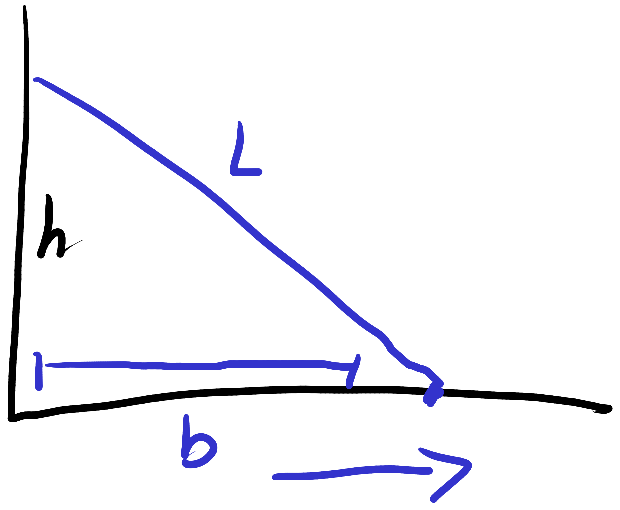
$\frac{d}{dt}$

$2D(t)\cdot D'(t) = 2A(t)\cdot A'(t) + 2B(t)\,B'(t)$

$100\ mi \cdot D'(2) = 80\ mi \cdot 40\ mi/h + 60\ mi \cdot 60\ mi/h$

$D'(2) = 8.4\ mi/h + 6.6\ mi/h = 68\ mi/h.$

Example 2.47. A twenty foot ladder rests against a wall. The bit on the wall is sliding down at 1 foot per second. How quickly is the bottom end sliding out when the top is 12 feet from the ground?



Q: $b' = ?$

$L = 20\text{ ft}$ $\quad L' = 0\text{ ft/s}$

$h = 12\text{ ft}$ $\quad h' = -1\text{ ft/s}$

$L^2 = h^2 + b^2 \Rightarrow 400\text{ ft}^2 = 144\text{ ft}^2 + b^2 \Rightarrow b = 16\text{ ft}$

$2 \cdot L \cdot L' = 2hh' + 2bb'$

~~$2 \cdot 20\text{ ft} \cdot 0\text{ ft/s}$~~ $= \cancel{2} \cdot 12\cancel{\text{ft}} \cdot -1\text{ ft/s} + \cancel{2} \cdot \cancel{b}\,(16\cancel{\text{ft}}) \cdot b'$ $\Rightarrow 12\text{ ft/s} = 16\,b'$

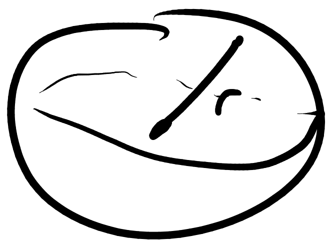
$24\text{ ft}^2/\text{s} = 32\text{ ft} \cdot b'$

$\frac{3}{4}\text{ ft/s} = b'$

Checklist

1) draw picture
2) whats Q?
3) Notation
4) Find eqn
5) Take derivative
6) plug in #s
7) Sanity Check

Example 2.48. A spherical balloon is inflating at 12 cm^3 per second. How quickly is the radius increasing when the radius is 3 cm?



Want $r' = \_\_\_ \text{ cm/s } (>0)$

$V' = 12 \text{ cm}^3/\text{s}$ $\quad V =$

$r' = ?$ $\quad r = 3\text{cm}$

Bonus fact

$V' = \underbrace{4\pi r^2}_{SA} r'$

$V = \frac{4}{3}\pi r^3$

$V' = \frac{4}{3}\pi 3r^2 \cdot r'$

$12 \text{ cm}^3/\text{s} = 4\pi 9 \text{cm}^2 \cdot r'$

$\frac{1}{3\pi} \text{ cm/s} = r'$.

$(V^2)' = 2V \cdot V'$

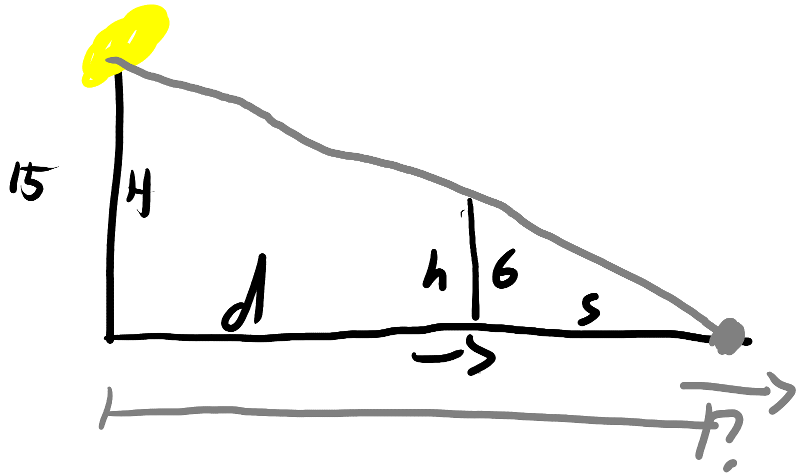
$(V)' = V'$

$\frac{d}{dt} V = \frac{dV}{dt}$

$\frac{d}{dt} V = 1 \cdot V^0 \cdot V'$

Example 2.51. A street light is mounted at the top of a 15-foot-tall pole. A six-foot-tall man walks straight away from the pole at 5 feet per second. How fast is the tip of his shadow moving when he is forty feet from the pole?

$$\left(\frac{d}{dt}\frac{s}{h} = \frac{s'h - h's}{h^2} = \frac{s'h}{h^2} = \frac{s'}{h}\right)$$



$H = 15$ ft

$h = 6$ ft

$d = 40$ ft $\qquad d' = 5$ ft/s

$s = \qquad\qquad s' =$

$$\frac{s}{h} = \frac{s+d}{H} \Rightarrow \frac{s}{6} = \frac{s+d}{15} \Rightarrow \frac{s'}{6} = \frac{s'+d'}{15}$$

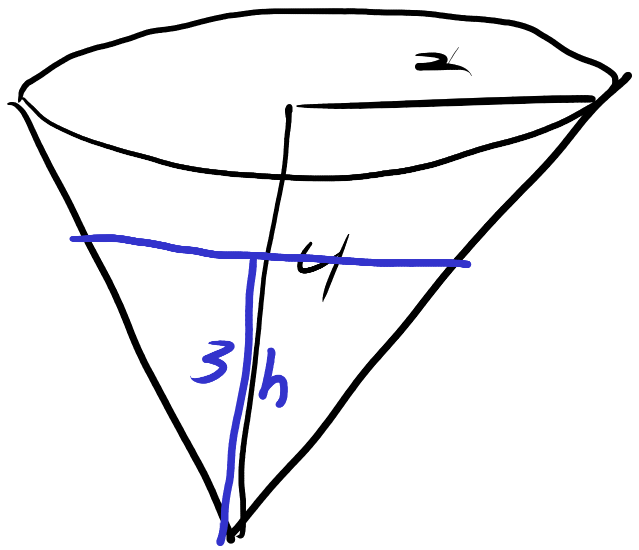
$$\frac{s'}{6\text{ ft}} = \frac{s' + 5\text{ ft/s}}{15\text{ ft}} \Rightarrow 15s' = 6s' + 30\text{ ft/s}$$

$$9s' = 30\text{ ft/s}$$

$$s' = \frac{10}{3}\text{ ft/s}$$

A: $s' + d' = \left(\frac{10}{3} + 5\right)$ ft/s $= \frac{25}{3}$ ft/s.

Example 2.50. An inverted conical water tank with radius 2m and height 4m is being filled with water at a rate of $2\text{m}^3/\text{min}$. How fast is the water rising when the water is 3 m tall?



Q: $h' =$ ___ m/min

$H = 4m \quad R = 2m$

$V' = 2\,m^3/min \quad V =$

$h = 3m \quad h' = ?$

$V = \frac{1}{3}\pi r^2 h$.

$\frac{R}{H} = \frac{r}{h} \Rightarrow r = \frac{h}{2}$

$r' = h'/2$

1) $V' = \frac{1}{3}\pi(2rr'h + r^2h')$

$2\frac{m^3}{m} = \frac{\pi}{3}\left(2\cdot\frac{3}{2}m\cdot\frac{h'}{2}\cdot 3m + (3/2\,m)^2 h'\right)$

$2\,m^3/min = \frac{\pi}{3}\left(\frac{9}{2}h'm^2 + \frac{9}{4}h'm^2\right)$

$\frac{6}{\pi}\,m/min = \frac{27}{4}h'$

$h' = \frac{8}{9\pi}\,m/min.$

2) $V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$

$V' = \frac{\pi}{4}h^2 h'$

$2\,m^3/min = \frac{\pi}{4}\cdot 9\,m^2\cdot h'$

$\frac{8}{9\pi}\,m/min = h'$.