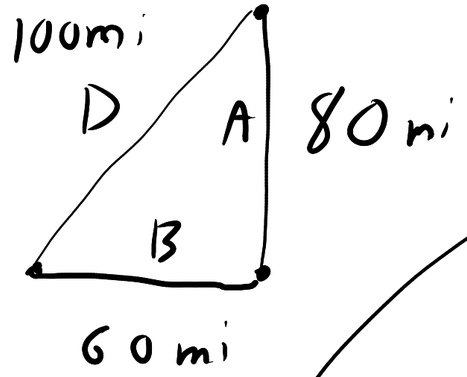
Related Rates

derivative - rate of change

Stocks/levels versus flows/rates

If you have a relationship on levels
then can find a relationship
of rates

Example 2.46. Suppose one car drives north at 40 mph, and an hour later another starts driving west from the same place at 60 mph. After a second hour, how quicly is the distance between them increasing?



$A(2) = 80$ mi $\quad A'(2) = 40$ mi/h

$B(2) = 60$ mi $\quad B'(2) = 60$ mi/h

Q: $D'(2) = ?$

$$D(t)^2 = A(t)^2 + B(t)^2$$

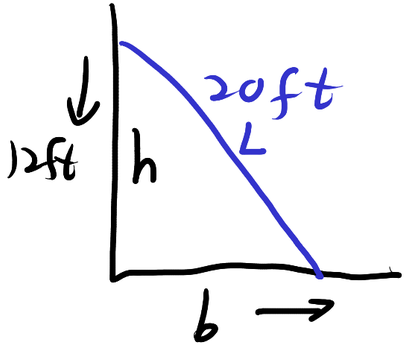
$$D(2) = \sqrt{A(2)^2 + B(2)^2} = \sqrt{80^2 + 60^2} = \sqrt{10{,}000} = 100 \text{ mi}$$

$$2\,D(t) \cdot D'(t) = 2\,A(t) \cdot A'(t) + 2\,B(t)\,B'(t)$$

$$100 \text{ mi} \cdot D'(2) = 80 \text{ mi} \cdot 40 \text{ mi/h} + 60 \text{ mi} \cdot 60 \text{ mi/h}$$

$$D'(2) = 8 \cdot 4 \text{ mi/h} + 6 \cdot 6 \text{ mi/h} = \boxed{68 \text{ mi/h}}.$$

Example 2.47. A twenty foot ladder rests against a wall. The bit on the wall is sliding down at 1 foot per second. How quickly is the bottom end sliding out when the top is 12 feet from the ground?



$h = 12\text{ ft}$ $\quad h' = -1\text{ ft/s}$

$b = 16\text{ ft}$ $\quad b' = ?$

$L = 20\text{ ft}$ $\quad L' = 0\text{ ft/s}$

$$b^2 + h^2 = L^2$$

$$2b \cdot b' + 2h h' = 2L L'$$

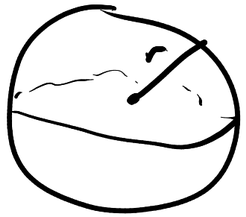
$$2 \cdot 16\text{ ft} \cdot b' + 2 \cdot 12\text{ ft} \cdot (-1\text{ ft/s}) = 0$$

$$16\, b' = 12\text{ ft/s} \Rightarrow \boxed{b' = 3/4\text{ ft/s}.}$$

Checklist

1) Draw pic
2) What should A look like?
3) Notation
4) write down eqn
5) diff eqn
6) plug in done
7) sanity check

Example 2.48. A spherical balloon is inflating at 12 cm^3 per second. How quickly is the radius increasing when the radius is 3 cm?



$r = 3\,cm \qquad r' = ?$

$V = \qquad V' = 12\,cm^3/s$

$$V = \frac{4}{3}\pi r^3$$

$$V' = \frac{4}{3}\pi 3r^2 r' \rightarrow \boxed{4\pi r^2 r'}$$

$4\pi r^2$ ← SA sphere

$$12\,cm^3/s = 4\pi \cdot 9\,cm^2 \cdot r'$$

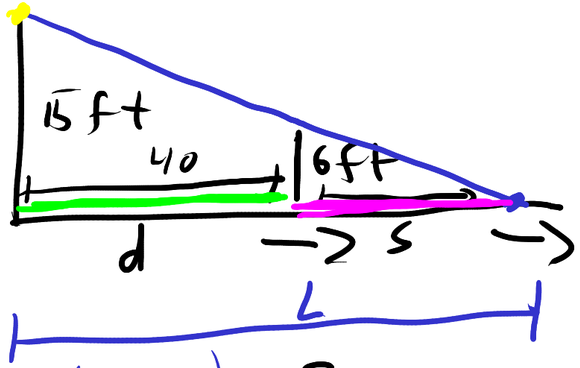
$$12\,cm/s = 36\pi r'$$

$$r' = \boxed{\frac{1}{3\pi}\,cm/s}.$$

Example 2.51. A street light is mounted at the top of a 15-foot-tall pole. A six-foot-tall man walks straight away from the pole at 5 feet per second. How fast is the tip of his shadow moving when he is forty feet from the pole?

Q: $d'_{ts} = ?$

$(d_{ts})' = ?$



$d = 40$ ft $\quad d' = 5$ ft/s

$m = 6$ ft $\quad m' = 0$

$L = 15$ ft $\quad L' = 0$

$s =$ $\quad s' =$

$d'_{ts} = 5 \text{ ft/s} + \frac{10}{3} \text{ ft/s}$

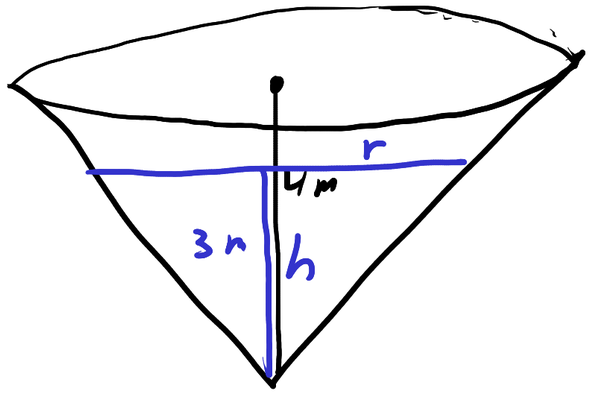
$= \frac{25}{3}$ ft/s.

$\frac{L-d}{6} = \frac{L}{15}$

$\frac{s}{6\text{ ft}} = \frac{s+d}{15\text{ ft}} \Rightarrow \frac{s'}{6\text{ ft}} = \frac{s'+d'}{15\text{ ft}} \Rightarrow 15s' = 6s' + 6d'$

$\Rightarrow 9s' = 6d' \Rightarrow s' = \frac{2}{3}d' = \frac{10}{3}$ ft/s.

Example 2.50. An inverted conical water tank with radius 2m and height 4m is being filled with water at a rate of $2\text{m}^3/\text{min}$. How fast is the water rising when the water is 3 m tall?



Q: $h' = ?$ m/min $h = 3m$

$V' = 2\ m^3/min$ $V =$

$r' =$ $r = 3/2\ m$

$H = 4m$ $R = 2m$

$\frac{r}{h} = \frac{R}{H} = \frac{1}{2}$

$r = h/2 \Rightarrow r' = h'/2$

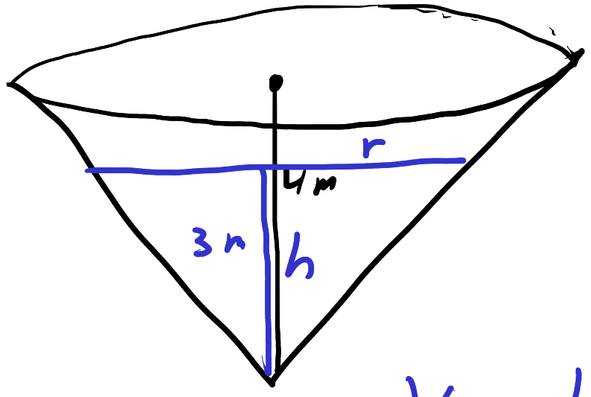
$V = \frac{1}{3}\pi r^2 h$

$V' = \frac{1}{3}\pi (2rr'h + r^2 h')$

$2\ m^3/min = \frac{\pi}{3}\left(2 \cdot \frac{3}{2}m \cdot \frac{h'}{2} \cdot 3m + (3/2\,m)^2 \cdot h'\right) = \frac{\pi}{3}\left(\frac{9}{2}m^2 h' + \frac{9}{4}m^2 h'\right)$

$2\ m^3/min = \frac{9\pi}{4} m^2 h' \Rightarrow \boxed{h' = \frac{8}{9\pi}\ m/min}$

Example 2.50. An inverted conical water tank with radius 2m and height 4m is being filled with water at a rate of $2\text{m}^3/\text{min}$. How fast is the water rising when the water is 3 m tall?



Q: $h' = ?$ m/min $h = 3m$

$V' = 2\ m^3/min$ $V =$

$r' =$ $r = 3/2\ m$

$H = 4m$ $R = 2m$

$\frac{r}{h} = \frac{R}{H} = \frac{1}{2}$

$r = h/2$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$V = \frac{1}{12}\pi h^3$

$V' = \frac{\pi}{12} 3h^2 h'$

$2\ m^3/min = \frac{\pi}{4}(3m)^2 h' = \frac{9\pi}{4}\ m^2\ h'$

$\frac{8}{9\pi}\ m/min = h'$