Math 1231 Midterm Solutions

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1. This test is due Tuesday at midnight. Logistically, this will work just like the mastery quizzes: download it, write up your answers, and upload them to Blackboard for us to grade.
2. You will have two hours for this test. Please write down your start and end times on the test and include that in your upload. You may not spend more than two hours on the test unless you have a specific accommodation.
3. You are not allowed to consult books or notes during the test, but you may use a one-page cheat sheet you have made for yourself ahead of time. Please upload your sheet along with your test.
4. If you have questions, I will be online and responsive during the usual class times. If you want to take the test at a time you know I'll be able to answer any questions quickly, I encourage you to use one of those time slots.
5. You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.

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Problem 1. Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 4} \frac{x^2-6x+8}{x^2-3x-4}$$

Solution:

$$\lim_{x\to 4} \frac{x^2-8x+6}{x-4} = \lim_{x\to 4} \frac{(x-4)(x-2)}{(x-4)(x+1)} = \lim_{x\to 4} \frac{x-2}{x+1} = 2/5.$$

(b)

$$\lim_{x\to 2} \frac{x^2+x+1}{(x-2)^2}$$

Solution: The limit of the top is 7 and the limit of the bottom is 0, so this limit is $\pm\infty$. In fact the top and bottom are both (always) positive, so the limit is $+\infty$.

(c)

$$\lim_{x\to +\infty} \frac{\sqrt{4x^2+1}}{x+5}$$

Solution:

$$\lim_{x\to +\infty} \frac{\sqrt{4x^2+1}}{x+5} = \lim_{x\to +\infty} \frac{\sqrt{4+1/x^2}}{1+5/x} = \frac{\sqrt{4+0}}{1+0} = 2.$$

(d)

$$\lim_{x\to -3} \frac{\sqrt{10+2x}-2}{x+3}$$

Solution:

$$\begin{aligned}
\lim_{x\to -3} \frac{\sqrt{10+2x}-2}{x+3} &= \lim_{x\to -3} \frac{10+2x-4}{(x+3)(\sqrt{10+2x}+2)} \\
&= \lim_{x\to -3} \frac{2(x+3)}{(x+3)(\sqrt{10+2x}+2)} \\
&= \lim_{x\to -3} \frac{2}{\sqrt{10+2x}+2} = \frac{2}{4} = 1/2.
\end{aligned}$$

Problem 2.

(a) **Directly from the definition of derivative,** compute the derivative of $f(x) = \frac{1}{x+2}$ at $a = 3$.

Solution:

$$\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h} \\
&= \lim_{h\to 0} \frac{5-(5+h)}{5h(5+h)} \\
&= \lim_{h\to 0} \frac{-1}{5(5+h)} = \frac{-1}{25}.
\end{aligned}$$

(b) The force a magnet exerts on a piece of iron depends on the distance between the magnet and the metal. Let $F(d) = \frac{2}{d^2}$ give the force exerted by the magnet in Newtons, where d is the distance between them in meters.

(i) What does the derivative $F'(d)$ represent, and what are its units?

Solution: The derivative is the rate at which the amount of force changes as you change the distance between the magnet and the iron; its units are Netwons per meter.

(ii) Calculate $F'(2)$. What does this tell you physically?

Solution: $F'(d) = \frac{-4}{d^3}$ so $F'(2) = \frac{-4}{8} = -1/2$. This means that moving the iron another meter away from the magnet should reduce the force by about half a Newton.

Problem 3.

(a) Find an equation of the line tangent to $y = x\sqrt{x^3+1}$ at the point $(2, 6)$.

Solution: We have

$$y' = \sqrt{x^3+1} + x\frac{1}{2}(x^3+1)^{-1/2} \cdot 3x^2$$
$$y'(2) = 3 + 2 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot 12 = 7.$$

so the equation of the tangent line is

$$y - 6 = 7(x-2)$$

or

$$y = 6 + 7(x-2) = 7x - 8.$$

(b) Give an equation for the linear approximation of the function $f(x) = \sqrt[3]{4x+3}$ near the point $a = 6$, and use it to estimate $f(6.1)$.

Solution: We calculate that $f(6) = \sqrt[3]{27} = 3$, and

$$f'(x) = \frac{1}{3}(4x+3)^{-2/3} \cdot 4$$
$$f'(6) = \frac{4}{3}27^{-2/3} = \frac{4}{27}$$

so our approximation equation is

$$f(x) \approx 3 + \frac{4}{27}(x-6).$$

We estimate that

$$f(6.1) \approx 3 + \frac{4}{27}(6.1-6) = 3 + \frac{4}{270} = \frac{407}{135} \approx 3.0148.$$

Problem 4. Compute the derivatives of the following functions using any methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \tan\left(\frac{x^3 + \sqrt[3]{x}}{\sin(x^2+1)}\right)$

Solution:

$$f'(x) = \sec^2\left(\frac{x^3 + \sqrt[3]{x}}{\sin(x^2+1)}\right) \frac{(3x^2 + \frac{1}{3}x^{-2/3})\sin(x^2+1) - \cos(x^2+1)2x(x^3+\sqrt[3]{x})}{\sin^2(x^2+1)}$$

(b) $g(x) = \sin\left(\sec^2\left(\sqrt{x^3+x}\right)\right)$

Solution:

$$g'(x) = \cos\left(\sec^2\left(\sqrt{x^3+x}\right)\right) \cdot 2\sec\left(\sqrt{x^3+x}\right)\sec\left(\sqrt{x^3+x}\right)\tan\left(\sqrt{x^3+x}\right)\frac{1}{2}(x^3+x)^{-1/2}(3x^2+1).$$

Problem 5. (a) A curve is defined by the equation $x^3y - 10 + xy^2 = 0$.

(i) Verify that the curve passes through the point $(2, 1)$.

(ii) What is the equation of the line that is tangent to the curve at the point $(2, 1)$?

Solution:

(i) $2^3 \cdot 1 - 10 + 2 \cdot 1^2 = 8 - 10 + 2 = 0$.

(ii) We compute

$$\begin{aligned} 3x^2y + x^3y' + y^2 + 2xyy' &= 0 \\ 12 + 8y' + 1 + 4y' &= 0 \\ 12y' &= -13 \\ y' &= -13/12. \end{aligned}$$

Alternatively, we could do this the long way:

$$\begin{aligned} 3x^2y + x^3y' + y^2 + 2xyy' &= 0 \\ (x^3 + 2xy)y' &= -3x^2y - y^2 \\ y' &= -\frac{3x^2y + y^2}{x^3 + 2xy} \\ y' &= -\frac{12+1}{8+4} = -13/12. \end{aligned}$$

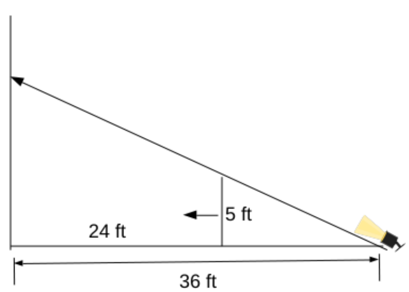
Eitehr way, we get the equation

$$y - 1 = -13/12(x - 2).$$

or

$$y = -13/12x + 19/6.$$

(b) A spot light is on the ground 36 ft away from a wall and a 5 ft tall person is walking towards the wall at a rate of 4 ft/sec. How fast is the height of the shadow changing when the person is 24 feet from the wall? Is the shadow increasing or decreasing in height at this time?



Solution: Let h be the height of the shadow, and d be the distance between the wall and the person. Then we want to find h'. We currently have $d = 24$. We know by similar triangles that $\frac{36-d}{36} = \frac{5}{h}$, which tells us that currently $h = 15$.

Then we have $d' = -4$. We compute

$$\begin{aligned}
\frac{-d'}{36} &= \frac{-5h'}{h^2} \\
\frac{1}{9} &= \frac{-h'}{45} \\
h' &= \frac{-45}{9} = -5.
\end{aligned}$$

Thus the shadow's height is decreasing by 5 feet per second.