Math 1231 Final Solutions

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1. This test is due at the scheduled exam time. Logistically, this will work just like the mastery quizzes: download it, write up your answers, and upload them to Blackboard for us to grade.
2. You will have two hours for this test. Please write down your start and end times on the test and include that in your upload. You may not spend more than two hours on the test unless you have a specific accommodation.
3. You are not allowed to consult books or notes during the test, but you may use a one-page cheat sheet you have made for yourself ahead of time. Please upload your sheet along with your test.
4. If you have questions, I will be online and responsive during the scheduled exam time. If you want to take the test at a time you know I'll be able to answer any questions quickly, I encourage you to use that time slot.
5. You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.

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Problem 1. (a) A curve is defined by the equation $x^4 - 2x^2y^2 + y^4 = 16$. Verify that the curve passes through the point $(\sqrt{5}, 1)$. What is the equation of the tangent line to the curve at this point?

Solution: If we plug in $\sqrt{5}$ for x and 1 for y we get $25 - 2 \cdot 5 \cdot 1 + 1 = 16$, so the point $(\sqrt{5}, 1)$ is on the curve.

To find the tangent line, we use implicit differentiation, and find that

$$4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$

$$4x^3 - 4xy^2 = 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}$$

$$\frac{4x^3 - 4xy^2}{4x^2y - 4y^3} = \frac{dy}{dx}$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5} \cdot 1^2}{4\sqrt{5}^2 \cdot 1 - 4 \cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$y - y_0 = m(x - x_0)$$
$$y - 1 = \sqrt{5}(x - \sqrt{5}).$$

(b) Find all the critical points of $g(x) = \dfrac{x^2 - 8}{x + 3}$

Solution: The function is undefined at $x = -3$.

$g'(x) = \frac{2x(x+3) - 1(x^2-8)}{(x+3)^2} = \frac{x^2+6x+8}{(x+3)^2}$. The denominator is zero when $x = -3$, and thus the derivative is undefined there, but so is the function, so we can count this as a critical point or not, to our taste. The numerator is $(x + 2)(x + 4)$ and thus has roots when $x = -2, -4$. So the critical points of the function are -2 and -4, and possibly -3.

(c) Find the absolute extrema of $f(x) = 3x^4 - 20x^3 + 24x^2 + 7$ on $[0, 5]$.

Solution: f is a continuous function on a closed interval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 12x^3 - 60x^2 + 48x = 12x(x^2 - 5x + 4) = 12x(x - 4)(x - 1)$ is defined everywhere and has roots at $0, 1, 4$. The endpoints are $0, 5$, so we need to evaluate f at $0, 1, 4, 5$.

$$f(0) = 7$$
$$f(1) = 14$$
$$f(4) = 3(4^4) - 5(4^4) + \frac{3}{2}(4^4) + 7 = \frac{-1}{2}4^4 + 7 = 7 - 128 = -121$$
$$f(5) = 3 \cdot 5^4 - 4 \cdot 5^4 + 5^4 - 5^2 + 7 = 7 - 25 = -18.$$

So the absolute maximum is 14 at 1, and the absolute minimum is -121 at 4.

Problem 2. (a) Classify the relative extrema of $h(x) = \sqrt[3]{x}(x + 4)$

Solution: We have

$$h'(x) = \sqrt[3]{x} + \frac{1}{3}x^{-2/3}(x + 4) = \frac{x}{\sqrt[3]{x^2}} + \frac{x + 4}{3\sqrt[3]{x^2}} = \frac{4x + 4}{3\sqrt[3]{x^2}}$$

so $h'(x)$ is undefined at $x = 0$ and $h'(x) = 0$ at $x = -1$. Thus the critical points are $0, -1$. Those are the possible relative extrema.

We can classify these points in two ways. We can use the first derivative test or the second derivative test. In these solutions I'll do both.

For the second derivative test we compute:

$$\begin{aligned} h''(x) &= \frac{4(3\sqrt[3]{x^2}) - \frac{4}{3}(x+1)\frac{-2}{3}x^{-5/3}}{9\sqrt[3]{x^4}} = \frac{12\sqrt[3]{x^2} + \frac{8}{3}(x+1)x^{-5/3}}{9\sqrt[3]{x^4}} \\ h''(-1) &= \frac{12+0}{9} = \frac{4}{3} > 0 \\ h''(0)\text{“} &= \text{”} \frac{0+0}{0} \text{ is undefined} \end{aligned}$$

So we see that h has a local minimum at -1 since $h''(-1) > 0$, but this tells us nothing about the critical point at 0; the second derivative test is inconclusive there. So we're forced to use the first derivative test.

For the first derivative test we make a chart:

	$4x+4$	$\frac{1}{3\sqrt[3]{x^2}}$	$h'(x)$
$x < -1$	$-$	$+$	$-$
$-1 < x < 0$	$+$	$+$	$+$
$0 < x$	$+$	$+$	$+$

so h has a relative minimum at -1 and neither a maximum nor a minimum at 0.

(The first derivative test was definitely the easier path here).

(b) Ten miles from home you remember that you left the water running, which is costing you 90 cents an hour. Driving home at speed s miles per hour costs you $4(s/10)$ cents per mile. At what speed should you drive to minimize the total cost of gas and water?

Solution: The water will be running for $10/s$ hours and thus the total cost of water will be $900/s$ cents. The cost of driving will be $10 \cdot 4(s/10) = 4s$ cents. Thus our total cost is $C(s) = 4s + 900/s$, and we want to minimize this.

We have $C'(s) = 4 - 900/s^2$. This has critical points at $s = 0$ and when $4s^2 = 900$ and thus $s^2 = 225$ and $s = \pm 15$. Clearly we must have $s > 0$ for physical reasons, so the only relevant critical point is $s = 15$.

Checking the second derivative we have $C''(s) = 1800/s^3$ and thus $C''(15) = 8/15 > 0$ and thus $s = 15$ is a local minimum. In fact s is the global minimum for positive values; we can see this since $C'(s) < 0$ when $0 < s < 15$ and $C'(s) > 0$ when $s > 15$. Thus you should drive at 15 miles per hour.

Problem 3. (a) If $g(x) = \cos(x)$, use a quadratic approximation centered at 0 to estimate $g(.1)$.

Solution: We have $g'(x) = -\sin(x)$ and $g''(x) = -\cos(x)$. So $g'(0) = 0$ and $g''(0) = -1$, and then we have

$$\begin{aligned} g(x) &\approx g(0) + g'(0)(x-0) + \frac{g''(0)}{2}(x-0)^2 = 1 + 0x - \frac{1}{2}x^2 = 1 - x^2/2 \\ g(.1) &\approx 1 - .1^2/2 = .995. \end{aligned}$$

(b) Use two iterations of Newton's Method starting at 2 to estimate $\sqrt[3]{7}$.

Solution:

We start with $x_0 = 2$. We need to take $f(x) = x^3 - 7$, so $f'(x) = 3x^2$. Then

$$\begin{aligned} x_1 &= 2 - \frac{1}{12} = \frac{23}{12} \\ x_2 &= \frac{23}{12} - \frac{71/1728}{529/48} = \frac{18215}{9522} \approx 1.91294 \end{aligned}$$

(c) Using four rectangles and right endpoints, approximate the area under the curve $\sqrt{x}$ between $x = 5$ and $x = 9$.

Solution:

$$\begin{aligned} R_4 &= \sum_{i=1}^{4} \frac{4}{4}\sqrt{5 + 4\frac{i}{4}} \\ &= \sqrt{6} + \sqrt{7} + \sqrt{8} + \sqrt{9}. \end{aligned}$$

Problem 4. (a) Using **only the definition of Riemann sum** and your knowledge of limits, compute the exact area under the curve $x^2 + x^3$ between $x = 1$ and $x = 3$.

Solution: We compute

$$\begin{aligned} R_n &= \sum_{i=1}^{n} \frac{2}{n} f\left(1 + \frac{2i}{n}\right) = \frac{2}{n}\sum_{i=1}^{n} (1 + 2i/n)^2 + (1 + 2i/n)^3 \\ &= \frac{2}{n}\sum_{i=1}^{n} (1 + 4i/n + 4i^2/n^2) + (1 + 6i/n + 12i^2/n^2 + 8i^3/n^3) \\ &= \frac{2}{n}\sum_{i=1}^{n} 2 + 10i/n + 16i^2/n^2 + 8i^3/n^3 \\ &= \frac{4}{n}\sum_{i=1}^{n} 1 + \frac{20}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3 \\ &= \frac{4}{n}\cdot n + \frac{20}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4} \\ \lim_{n\to+\infty} R_n &= \lim_{n\to+\infty} \frac{4}{n}\cdot n + \frac{20}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4} \\ &= 4 + 10 + \frac{32}{3} + 4 = \frac{86}{3}. \end{aligned}$$

(b) Let $G(x) = \int_1^{x^2+1} t\sqrt{1 - t^2}\, dt$. What is $G'(x)$?

Solution: $G'(x) = \sqrt{1 - (x^2+1)^2}(x^2+1)\cdot(x^2+1)' = \sqrt{1 - (x^2+1)^2}\cdot(x^2+1)\cdot 2x$

(c) Compute $\displaystyle\int \sin^4(t)\cos(t)\, dt$

Solution: We can take $u = \sin(t)$, then we have $du = \cos(t)\, dt$ so we are computing

$$\int u^4\, du = \frac{1}{5}u^5 + C = \frac{\sin^5(t)}{5} + C.$$

Problem 5. (a) **By explicitly changing the bounds of the integral**, compute $\displaystyle\int_0^4 x^3\sqrt{9 + x^2}\, dx$.

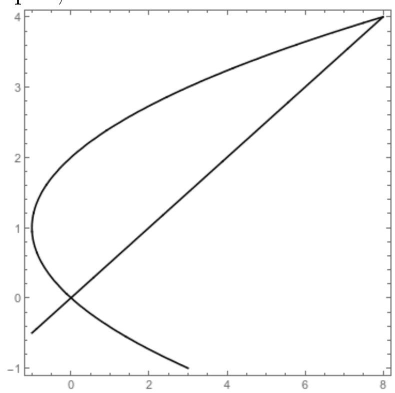
Solution: Take $u = 9 + x^2$ so that $x^2 = u - 9$ and $dx = u/2x$. Then $u(0) = 9$ and $u(4) = 25$ and we have

$$\begin{aligned} \int_0^4 x^3\sqrt{9 + x^2}\, dx &= \int_9^{25} x^2\sqrt{u}\frac{du}{2x} = \int_9^{25} \frac{1}{2}(u - 9)\sqrt{u}\, du \\ &= \frac{1}{2}\int_9^{25} u^{3/2} - 9u^{1/2}\, du \\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - 6u^{3/2}\right)\Bigg|_9^{25} = \frac{1}{2}(1250 - 750 - (486/5 - 162)) \\ &= \frac{1}{2}(662 - 486/5) = \frac{3310 - 486}{10} = \frac{2824}{10} = 1412/5. \end{aligned}$$

(b) Find the area of the region bounded by $x - 2y = 0$ and $x = y^2 - 2y$.

Solution:

We sketch the region, and see that it will be much easier to integrate with respect to y. Setting the two equations equal, we see the curves intersect when $y^2 - 4y = 0$, and thus when $y = 0, 4$, and thus at $(0,0)$



and $(8,4)$.

$$A = \int_0^4 2y - (y^2 - 2y)\, dy = \int_0^4 4y - y^2\, dy = 2y^2 - \frac{y^3}{3}|_0^4 = 32 - \frac{64}{3} = \frac{32}{3}.$$

(c) What is the average value of the function $h(x) = x^2 + x$ on the interval $[0,6]$?

Solution:

$$\begin{aligned} h_{ave} &= \frac{1}{6}\int_0^6 x^2 + x\, dx \\ &= \frac{1}{6}(x^3/3 + x^2/2)|_0^6 \\ &= \frac{1}{6}(72 + 18) = 15. \end{aligned}$$