Problem 1. Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x}$$

Solution:

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x} = \lim_{x\to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{9-x}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{1}{3+\sqrt{x}} = 1/6.$$

(b)

$$\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x}$$

Solution:

$$\begin{aligned}\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x} &= \lim_{x\to -\infty} \frac{3x^3/x^3+\sqrt[3]{x}/x^3}{\sqrt{9x^6+2x^2+1}/(-\sqrt{x^6})+x/x^3} \\ &= \lim_{x\to -\infty} \frac{3+x^{-8/3}}{-\sqrt{9+2x^{-4}+x^{-6}}+x^{-2}} \\ &= \lim_{x\to -\infty} \frac{3}{-\sqrt{9}} = -1.\end{aligned}$$

(c)

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} =$$

Solution:

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} = \lim_{x\to 1} \left(\frac{\sin(x-1)}{x-1}\right)^2 = \left(\lim_{x\to 1} \frac{\sin(x-1)}{x-1}\right)^2 = 1^2 = 1$$

by the small angle approximation.

(d)

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} =$$

Solution:

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} = -\infty$$

since the top approaches -2 and the bottom approaches zero and is always positive.

Problem 2.

(a) **Directly from the definition of derivative,** compute the derivative of $f(x) = x^2 + \sqrt{x}$ at $a = 2$.

Solution:

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)^2+\sqrt{2+h}-2^2-\sqrt{2}}{h} \\
&= \left(\lim_{h\to 0} \frac{4h+h^2}{h}\right) + \left(\lim_{h\to 0} \frac{(\sqrt{2+h}-\sqrt{2})(\sqrt{2+h}+\sqrt{2})}{h(\sqrt{2+h}+\sqrt{2})}\right) \\
&= \left(\lim_{h\to 0} 4+h\right) + \left(\lim_{h\to 0} \frac{1}{\sqrt{2+h}+\sqrt{2}}\right) \\
&= 4+\frac{1}{2\sqrt{2}}.
\end{aligned}$$

(b) Suppose that $Q(p) = 3p^2 + 10p - 100$ is the number of widgets you can buy at a price of p dollars.

(i) What does the derivative $Q'(p)$ represent, and what are its units?

Solution: The derivative is the rate at which increasing the price increases the number of widgets you can buy (called the marginal elasticity of demand, though you don't need to know that on the test). Its units are widgets per dollar.

(ii) Calculate $Q'(10)$. What does this tell you?

Solution: $Q'(p) = 6p + 10$ so $Q'(10) = 70$. This means that if you are buying widgets for $10, you can get approximately seventy more widgets if you raise your price ot $11.

Problem 3.

(a) Find an equation of the line tangent to $y = \frac{x^2-1}{x^2+1}$ at the point $(0, -1)$.

Solution: We have that

$$y' = \frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}$$

so when $x = 0$ we have $y' = (0-0)/1 = 0$. The equation for a tangent line is $y = m(x-x_0)+y_0$, so the tangent line to this function at $(0, 1)$ is $y = 0(x-0)+(-1)$, or $y = -1$.

(b) Give equation for the linear approximation of the function $f(x) = x\sin(x)$ near the point $a = \pi/2$.

Solution: We calculate that $f(x) = \pi/2\sin(\pi/2) = \pi/2$, and $f'(x) = \sin(x)+x\cos(x)$, so $f'(\pi/2) = \sin(\pi/2)+\pi/2\cos(\pi/2) = 1$. So

$$f(x) \approx \pi/2 + 1(x-\pi/2) = x.$$

Problem 4. Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right)$

Solution:

$$f'(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right)\cdot\tan\left(\frac{\sqrt{x^2+1}}{x+2}\right)\cdot\frac{\frac{1}{2}(x^2+1)^{-1/2}2x(x+2)-\sqrt{x^2+1}}{(x+2)^2}$$

(b) $g(x) = \sqrt[4]{\frac{x^3+\cos(x^2)}{\sin(x^3)+1}}$

Solution:

$$g'(x) = \frac{1}{4}\left(\frac{x^3+\cos(x^2)}{\sin(x^3)+1}\right)^{-3/4}\cdot\frac{(3x^2-\sin(x^2)2x)(\sin(x^3)+1)-\cos(x^3)3x^2(x^3+\cos(x^2))}{(\sin(x^3)+1)^2}$$

Problem 5. (a) Find a tangent line to the curve given by $x^4 - 2x^2y^2 + y^4 = 16$ at the point $(\sqrt{5}, 1)$.

Solution: We use implicit differentiation, and find that

$$\begin{aligned} 4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} &= 0 \\ 4x^3 - 4xy^2 &= 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx} \\ \frac{4x^3 - 4xy^2}{4x^2y - 4y^3} &= \frac{dy}{dx} \end{aligned}$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5}\cdot 1^2}{4\sqrt{5}^2 \cdot 1 - 4\cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= \sqrt{5}(x - \sqrt{5}). \end{aligned}$$

(b) The surface area of a cube is given by the formula $A = 6s^2$ where s is the length of a side. If the side lengths are increasing by 2 inches per second, how fast is the surface area increasing when the area is 54 square inches?

Solution: We have the data $A = 6s^2, A = 54, s' = 2$. We take a derivative and see that $A' = 12ss'$, so we need to find s. But when $A = 54$ we have

$$\begin{aligned} 54 &= 6s^2 \\ 9 &= s^2 \\ 3 &= s \end{aligned}$$

and thus

$$A' = 12ss' = 12 \cdot 3 \cdot 2 = 72$$

so the area is increasing at 72 square inches per second.