Math 1231 Fall 2020
Single-Variable Calculus I Mastery Quiz 10
Due midnight on Thursday, November 12

This week's mastery quiz has twelve topics. (Or at least it will some time soon!) **Do not answer all ten.** You may answer the first question on the newest topic, numbered sixteen, and *two* additional topics. You may pick two topic you have not yet demonstrated mastery on and answer the questions on that topic. (If you are retrying a topic, please complete the entire page.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 10-20 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics:

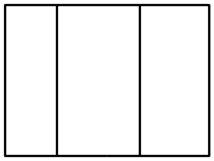
16. Optimization
15. Curve Sketching
14. First and Second Derivative Tests
13. Global Maxima and Critical Points
12. Related Rates
11. Implicit Differentiation
10. Rates of Change

To come:

6. Definition of a Derivative
5. Infinite Limits
4. Trigonometric Limits
3. Computing Limits
2. Formal limits

16. **Optimization**

We wish to build a rectangular pen with two parallel internal partitions, using 1000 feet of fencing. What dimensions maximize the total area of the pen?



Solution:

Our objective function is $A = \ell w$. We see also that $2\ell + 4w = 1000$ so we can write $\ell = 500 - 2w$, and thus we have

$$\begin{aligned} A &= (500 - 2w)w = 500w - 2w^2 \\ A' &= 500 - 4w \end{aligned}$$

has a critical point when $w = 125$.

We can see this is a maximum using the extreme value theorem: the function is defined on the interval $[0, 250]$, and $A(0) = A(250) = 0$.

Or we can use the first derivative test; we see that $A'(w) < 0$ when $w > 125$ and $A'(w) > 0$ when $w < 125$, so A has a local maximum at $w = 125$.

Or we can use the second derivative test. $A'' = -4 < 0$ so we have a local maximum.

Thus the pen is maximized with width 125 and length 250. (The maximum area, which I didn't ask for, is 31250 square feet.

15. **Curve Sketching**

Sketch the graph of $f(x) = x^5 - 5x^4 + 5x^3 = x^3(x^2 - 5x + 5)$. We have $f'(x) = 5x^2(x-3)(x-1)$ and $f''(x) = 10x(2x^2 - 6x + 3)$.

You should discuss the domain, limits, critical points, intervals of increase and decrease, concavity, and possible points of inflection.

Solution: The domain of f is all reals.

There are roots at 0 and at $5/2 \pm \sqrt{5}/2$.

We see that $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

The critical points are $0, 1, 3$. We compute $f(0) = 0, f(1) = 1, f(3) = -27$.

For increase and decrease we make a chart:

	$5x^2$	$(x-3)$	$(x-1)$	$f'(x)$
$x < 0$	$+$	$-$	$-$	$+$
$0 < x < 1$	$+$	$-$	$-$	$+$
$1 < x < 3$	$+$	$-$	$+$	$-$
$3 < x$	$+$	$+$	$+$	$+$

Thus f is increasing on $(-\infty, 1)$ and on $(3, +\infty)$, and is decreasing on $(1, 3)$.

The possible points of inflection are 0 and $\frac{6\pm\sqrt{36-24}}{4} = \frac{3\pm\sqrt{3}}{2}$. We can make a chart:

	$10x$	$2x^2-6x+3$	$f'(x)$
$x<0$	$-$	$+$	$-$
$0<x<(3-\sqrt{3})/2$	$+$	$+$	$+$
$(3-\sqrt{3})/2<x<(3+\sqrt{3}))/2$	$+$	$-$	$-$
$(3+\sqrt{3})/2<x$	$+$	$+$	$+$

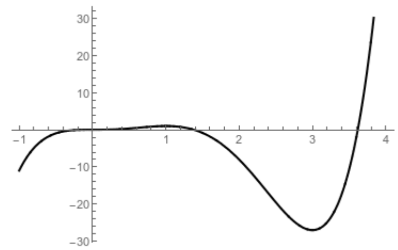


Figure 1: Graph of $f(x)$

14. **First and Second Derivative Tests**

(a) Classify all the critical points and relative extrema of $h(x) = x^3/(x+1)$. (For each critical point, tell me whether it is a relative maximum, a relative minimum, or neither.)

Solution:

We have

$$h'(x) = \frac{3x^2(x+1) - x^3}{(x+1)^2} = \frac{3x^3+3x^2-x^3}{(x+1)^2} = \frac{x^2(2x+3)}{(x+1)^2}$$

The critical points are thus at 0 and at $-3/2$, and a fake one at -1. We make a chart:

	x^2	$2x+3$	$(x+1)^{-2}$	$h'(x)$
$x<-3/2$	$+$	$-$	$+$	$-$
$-3/2<x<-1$	$+$	$+$	$+$	$+$
$-1<x<0$	$+$	$+$	$+$	$+$
$0<x$	$+$	$+$	$+$	$+$

This tells us that we have a local minimum at $x = -3/2$, and no other extrema. We compute $h(-3/2) = -27/8/(-1/2) = 27/4$, so the sole local minimum is $(-3/2, 27/4)$.

Alternatively we could use the second derivative test.

(b) Classify the critical points and relative extrema of $h(x) = \sin(x) + \cos(x)$ on $[0, 2\pi]$.

Solution: We have

$$h'(x) = \cos(x) - \sin(x)$$

so $h'(x)$ is defined everywhere, and has critical points where $\cos(x) = \sin(x)$. This happens when $x = \pi/4, 5\pi/4, 9\pi/4, \dots = \pi/4 + n\pi$. We only need to care about $\pi/4$ and $5\pi/4$.

We can classify these points in two ways. We can use the first derivative test or the second derivative test. In these solutions I'll do both.

For the second derivative test we compute:

$$
\begin{aligned}
h''(x) &= -\sin(x) - \cos(x) \\
h''(\pi/4) &= -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0 \\
h''(5\pi/4) &= \sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2} > 0.
\end{aligned}
$$

Thus h has a local maximum at $\pi/4$ and has a local minimum at $5\pi/4$.

For the first derivative test we make a chart:

	$h'(x)$
$0 < x < \pi/4$	$+$
$\pi/4 < x < 5\pi/4$	$-$
$5\pi/4 < x < 2\pi$	$+$

so h has a relative maximum at $\pi/4$ and a relative minimum at $5\pi/4$.

13. **Global Maxima and Critical Points**

(a) Find the absolute extrema of $f(x) = x^3 + x^2 - 5x$ on $[-1, 2]$.

Solution: f is a continuous function on a closed ineterval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 3x^2 + 2x - 5 = (3x + 5)(x - 1)$ is defined everywhere and has roots at $-5/3$ and 1, so the critical points are $-5/3, 1$. We can ignore $-5/3$ because it isn't in the interval, so we need to evaluate f at $-1, 1, 2$.

$$
\begin{aligned}
f(-1) &= 5 \\
f(1) &= -3 \\
f(2) &= 2
\end{aligned}
$$

So the absolute minimum is -3 at 1 , and the absolute maximum is 5 at -1.

(b) Find all the critical points of $g(x) = \sqrt[3]{x^2 - 3x + 2}$.

Solution: The function is defined everywhere. We compute

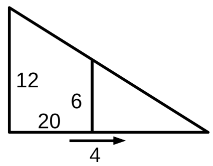
$$
g'(x) = \frac{1}{3}(x^2 - 3x + 2)^{-2/3}(2x - 3) = \frac{2x - 3}{3\sqrt{(x^2 - 3x + 2)^2}}.
$$

We have critical points where $g'(x) = 0$ and where it is undefined.

$g'(x) = 0$ precisely when $2x - 3 = 0$, precisely when $x = 3/2$. $g'(x)$ is undefined when $x^2 - 3x + 2 = 0$, so when $x = 1$ or $x = 2$. Thus g has critical points at $1, 3/2, 2$.

12. **Related Rates**

A street light is mounted at the top of a 12-foot-tall pole. A six-foot-tall man walks straight away from the pole at 4 feet per second. How fast is his shadow growing longer when he is twenty feet from the pole?



Solution: Let d be the distance of the man from the pole. Then $d = 20$ and $d' = 4$. If s is the length of the shadow, then we have $s/6 = (d+s)/12$ so we get

$$\begin{aligned} s &= \frac{d+s}{2} \\ s' &= d'/2 + s'/2 \\ s'/2 &= d'/2 \\ s' &= d' = 4. \end{aligned}$$

Thus the length of the shadow is growing at 4 feet per second.

11. **Implicit Differentiation**

(a) Write a tangent line to the curve $x^2y^2 = 5 + x + y$ at the point $(1, 3)$.

Solution: Implicit differentiation gives us

$$\begin{aligned} 2xy^2 + 2x^2yy' &= 1 + y' \\ 2 \cdot 1 \cdot 9 + 2 \cdot 1^2 \cdot 3 \cdot y' &= 1 + y' \\ 18 + 6y' &= 1 + y' \\ 5y' &= -17 \\ y' &= -17/5 \end{aligned}$$

and thus the tangent line has equation

$$y - 3 = \frac{-17}{5}(x - 1).$$

Alternatively we can compute

$$\begin{aligned} 2xy^2 + 2x^2yy' &= 1 + y' \\ y'(2x^2y - 1) &= 1 - 2xy^2 \\ y' &= \frac{1 - 2xy^2}{2x^2y - 1} \end{aligned}$$

(b) Find a formula for y' in terms of x and y if $xy^3 = \sqrt{x^2 + y^2}$.

Solution: Using implicit differentiation, we have

$$\begin{aligned} y^3 + 3xy^2y' &= \frac{2x + 2yy'}{2\sqrt{x^2+y^2}} \\ &= \frac{x + yy'}{\sqrt{x^2+y^2}} \\ y^3 - \frac{x}{\sqrt{x^2+y^2}} &= \frac{yy'}{\sqrt{x^2+y^2}} - 3xy^2y' \\ y' &= \frac{y^3 - \frac{x}{\sqrt{x^2+y^2}}}{\frac{y}{\sqrt{x^2+y^2}} - 3xy^2} \\ &= \frac{y^3\sqrt{x^2+y^2} - x}{y - 3xy^2\sqrt{x^2+y^2}}. \end{aligned}$$

10. **Rates of Change**

(a) Suppose that a factory produces widgets, and if p people work at the factory then they will produce a total of $W(p) = 30\sqrt{p}$ widgets.

(i) What does the derivative $W'(p)$ represent, and what are its units?
Solution: The derivative is the rate at which the number of widgets increases as we add more people to the factory (called the marginal product of labor). Its units are widgets per person.

(ii) Calculate $W'(9)$. What does this represent in the real world?
Solution: $W'(p) = \frac{15}{\sqrt{p}}$ so $W'(9) = 5$. So moving from nine people to ten people working at the factory will lead to the production of five extra widgets.

(b) Suppose the distance between two particles in centimeters is given as a function of time in seconds by the formula $d(t) = t^3 + 4t^2 + 5t + 4$.

(i) When is the velocity zero?
Solution: $d'(t) = 3t^2 + 8t + 5 = (3t+5)(t+1)$ so the velocity is zero when $t = -1, -5$.

(ii) When is the acceleration zero?
Solution: $d''(t) = 6t + 8$ is zero when $t = -4/3$.

6. **Definition of a Derivative**

Compute the following derivatives, *directly from the formal definition of derivative.*

(a) If $f(x) = \sqrt{2x+3}$, find $f'(3)$. **Solution:**

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{9+2h}-3}{h} \\ &= \lim_{h\to 0} \frac{2h}{h(\sqrt{9+2h}+3)} \\ &= \lim_{h\to 0} \frac{2}{\sqrt{9+2h}+3} \\ &= \frac{2}{6} = \frac{1}{3}. \end{aligned}$$

(b) If $g(x) = \frac{3}{x^2}$, find $g'(x)$. **Solution:**

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{(x+h)^2} - \frac{3}{x^2}}{h} \\ &= \lim_{h\to 0} \frac{3x^2 - 3(x^2+2xh+h^2)}{hx^2(x+h)^2} \\ &= \lim_{h\to 0} \frac{-6xh-3h^2}{hx^2(x+h)^2} \\ &= \lim_{h\to 0} \frac{-6x-3h}{x^2(x+h)^2} \\ &= \lim_{h\to 0} \frac{-6x}{x^4} = \frac{-6}{x^3}. \end{aligned}$$

5. **Infinite Limits** Compute:

(a) $\displaystyle\lim_{x\to+\infty} \frac{x(2x+3)(3x-2)}{x^3+x^2-3x+4} =$

Solution:

$$\lim_{x\to+\infty} \frac{x(2x+3)(3x-2)}{x^3+x^2-3x+4} = \lim_{x\to+\infty} \frac{(2+3/x)(3-2/x)}{1+1/x-3/x^2+4/x^3} = \frac{2\cdot 3}{1} = 6.$$

(b) $\displaystyle\lim_{x\to 2^+} \frac{x+1}{x-2} =$

Solution:

The limit of the top is 3 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the limit is from the right, the denominator is always positive, so the limit is $+\infty$.

(c) $\displaystyle\lim_{x\to 1} \frac{x-5}{x-1} =$

Solution: The limit of the top is -4 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the bottom can be positive or negative, we can't do better than that.

4. **Trigonometric Limits**

(a) Show that $\lim_{x\to 1}(x-1)^2\left(1+\sin\left(\frac{2}{x-1}\right)\right) = 0$.

Solution: We know that

$$\begin{aligned} -1 \le \sin\left(\frac{2}{x-1}\right) &\le 1 \\ 0 \le 1+\sin\left(\frac{2}{x-1}\right) &\le 2 \\ 0 \le (x-1)^2\left(1+\sin\left(\frac{2}{x-1}\right)\right) &\le 2(x-1)^2 \end{aligned}$$

Since $\lim_{x\to 1} 0 = 0$ and $\lim_{x\to 1} 2(x-1)^2 = 0$, by the Squeeze Theorem, we know that $\lim_{x\to 1}(x-1)^2\left(1+\sin\left(\frac{2}{x-1}\right)\right) = 0$.

(b) Compute $\lim_{x\to 3}\frac{\sin(x-3)\sin(6x-18)}{(x-3)^2}$.

Solution:

$$\lim_{x\to 3}\frac{\sin(x-3)\sin(6x-18)}{(x-3)^2} = \lim_{x\to 3}\frac{\sin(x-3)}{x-3}\frac{6\sin(6x-18)}{6x-18} = 1\cdot 6 = 6.$$

3. **Computing Limits** Compute:

(a) $\lim_{x\to 7}\frac{\sqrt{9+x}-4}{x-7} =$ **Solution:**

$$\lim_{x\to 7}\frac{\sqrt{9+x}-4}{x-7} = \lim_{x\to 7}\frac{9+x-16}{(x-7)(\sqrt{9+x}+4)} = \lim_{x\to 7}\frac{1}{\sqrt{9+x}+4} = \frac{1}{8}.$$

(b) $\lim_{x\to -2}\frac{1}{x+2}+\frac{1}{x^2+3x+2}$

Solution:

$$\begin{aligned} \lim_{x\to -2}\frac{1}{x+2}+\frac{1}{x^2+3x+2} &= \lim_{x\to -2}\frac{x^2+3x+2+x+2}{(x^2+3x+2)(x+2)} \\ &= \lim_{x\to -2} x^2+4x+4(x+2)^2(x+1) \\ &= \lim_{x\to -2} 1x+1 = -1. \end{aligned}$$

(c) $\lim_{x\to 1}\frac{x^2-5x+4}{x^2+6x-7} =$

Solution:

$$\lim_{x\to 1}\frac{x^2-5x+4}{x^2+6x-7} = \lim_{x\to 1}\frac{(x-4)(x-1)}{(x+7)(x-1)} = \lim_{x\to 1}\frac{x-4}{x+7} = \frac{-3}{8}.$$

2. **Formal Limits**

(a) Write a formal ϵ-δ proof that $\lim_{x\to 2} 2x + 2 = 4$.

Solution: Let $\epsilon > 0$ and set $\delta = \epsilon/2$. Then if $0 < |x - 2| < \delta$, we have

$$|2x + 2 - 4| = |2x - 2| = 2|x - 2| < 2\delta = \epsilon.$$

(b) Explicitly naming each limit law you use, compute

$\lim\limits_{x\to 3} \dfrac{3x^2+5}{4x-2} =$

Solution:

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x^2+3}}{2x+3} &= \frac{\lim_{x\to -1} \sqrt{x^2+3}}{\lim_{x\to -1} 2x+3} && \text{Quotients}\\
&= \frac{\sqrt{\lim_{x\to -1} x^2+3}}{\lim_{x\to -1} 2x+3} && \text{Exponents}\\
&= \frac{\sqrt{\lim_{x\to -1} x^2+\lim_{x\to -1} 3}}{\lim_{x\to -1} 2x+\lim_{x\to -1} 3} && \text{Sums}\\
&= \frac{\sqrt{\lim_{x\to -1} x^2+\lim_{x\to -1} 3}}{2\lim_{x\to -1} x+\lim_{x\to -1} 3} && \text{scalar products}\\
& && \text{[could also use regular product]}\\
&= \frac{\sqrt{(\lim_{x\to -1} x)^2+\lim_{x\to -1} 3}}{2\lim_{x\to -1} x+\lim_{x\to -1} 3} && \text{Exponents}\\
&= \frac{\sqrt{(\lim_{x\to -1} x)^2+3}}{2\lim_{x\to -1} x+3} && \text{Constants}\\
&= \frac{\sqrt{(-1)^2+3}}{2(-1)+3} = \frac{\sqrt{4}}{1} = 2 && \text{Identity}
\end{aligned}$$