Math 1231 Fall 2020
Single-Variable Calculus I Mastery Quiz 11
Due midnight on Thursday, November 19

This week's mastery quiz has seven topics. **Do not answer all five.** You may answer the two newest topics, numbered 18 and 17, and *one* additional topics. You may pick one topic you have not yet demonstrated mastery on and answer the questions on that topic. (If you are retrying a topic, please complete the entire page.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 10-20 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics:

18. Area and Riemann Sums
17. Approximation
16. Optimization
15. Curve Sketching
14. First and Second Derivative Tests
13. Global Maxima and Critical Points
9. Linear Approximations and Tangent Lines

18. **Area and Riemann Sums**

Let $f(x) = x^2 - x$ be defined on the interval $[-3, 0]$.

(a) Approximate the area under the curve of the function using three rectangles and right endpoints.

(b) Approximate the area under the curve of the function using three rectangles and left endpoints.

(c) Find a formula for computing R_n, the estimate using n rectangles and right endpoints. (This formula should not have a summation sign or be given as a sum of n terms.)

(d) Use the formula in part (c) to compute the area exactly.

Solution:

(a) $R_3 = 1 \cdot f(-2) + 1 \cdot f(-1) + 1 \cdot f(0) = 6 + 2 + 0 = 8.$

(b) $L_3 = 1 \cdot f(-3) + 1 \cdot f(-2) + 1 \cdot f(-1) = 12 + 6 + 2 = 20.$

(c)

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} \frac{3}{n} f\left(-3 + i\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{3}{n}\left((3i/n - 3)^2 - (3i/n - 3)\right) \\
&= \sum_{i=1}^{n} \frac{3}{n}\left(\frac{9i^2}{n^2} - \frac{18i}{n} + 9 - \frac{3i}{n} + 3\right) \\
&= \sum_{i=1}^{n} \frac{27i^2}{n^3} - \frac{63i}{n^2} + \frac{36}{n} \\
&= \frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{63}{n^2}\sum_{i=1}^{n} i + \frac{36}{n}\sum_{i=1}^{n} 1 \\
&= \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{63}{n^2}\frac{n(n+1)}{2} + \frac{36}{n}n
\end{aligned}$$

(d) We can compute

$$\begin{aligned}
\lim_{n\to+\infty} R_n &= \lim_{n\to+\infty} \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{63}{n^2}\frac{n(n+1)}{2} + \frac{36}{n}n \\
&= \lim_{n\to+\infty} \frac{27 \cdot 1(1+1/n)(2+1/n)}{6} - \frac{63 \cdot 1(1+1/n)}{2} + 36 \\
&= 9 - \frac{63}{2} + 36 = \frac{27}{2}.
\end{aligned}$$

17. **Approximation**

(a) Find a formula for the quadratic approximation of $f(x) = \sqrt{3x+1}$ near the point $a = 1$, and use it to estimate $f(1.01)$.

Solution: We compute

$$\begin{aligned} & & f(1) &= 2 \\ f'(x) &= \frac{3}{2}(3x+1)^{-1/2} & f'(1) &= \frac{3}{4} \\ f''(x) &= \frac{-9}{4}(3x+1)^{-3/2} & f''(1) &= \frac{-9}{32} \end{aligned}$$

and thus we have

$$\begin{aligned} f(x) &\approx 2 + \frac{3}{4}(x-1) - \frac{9}{64}(x-1)^2 \\ f(1.01) &\approx 2 + \frac{.03}{4} - \frac{.09}{64} = \frac{12839}{6400} = 2.00609375. \end{aligned}$$

(b) Use two steps of Newton's method to estimate $\sqrt{8}$ starting from $x_0 = 3$. (You should compute x_2.)

Solution: We're looking for a root of $g(x) = x^2 - 8$. So we compute $g'(x) = 2x$, and we get

$$\begin{aligned} x_1 &= 3 - \frac{g(3)}{g'(3)} = 3 - \frac{1}{6} = \frac{17}{6} \\ x_2 &= \frac{17}{6} - \frac{g(17/6)}{g'(17/6)} = \frac{17}{6} - \frac{289/36 - 288/36}{17/3} = \frac{17}{6} - \frac{1}{12 \cdot 17} = \frac{577}{204} \approx 2.8284 \end{aligned}$$

(The true answer is approximately 2.8284, so yay for us.)

16. **Optimization**

Suppose that a company that produces and sells x units of a product makes a revenue of $R(x) = 260x - 9x^2/10$ and has costs given by $C(x) = 1000 + 100x + x^2/10$. What is the maximum profit that can be made (where profit is revenues minus costs)?

Solution: Our profit function is $P(x) = R(x) - C(x) = -1000 + 160x - x^2$. Then

$$\begin{aligned} P'(x)&160 - 2x \\ 160 &= 2x \\ 80 &= x \end{aligned}$$

We can check that this is truly a maximum by the second derivative: $P''(x) = -2 < 0$ so we have a local maximum.

Or we can see that $P'(x) > 0$ when $x < 80$ and $P'(x) < 0$ when $x > 80$, so the function is increasing until 80 and decreasing after.

The profit at this quantity is

$$P(80) = -1000 + 160(80) - (80)^2 = -1000 + 12800 - 6400 = 5400.$$

15. **Curve Sketching**

Sketch the graph of $g(x) = 3x^4 - 4x^3 - 36x^2 + 64 = (x+2)^2(3x-4)(x-4)$ have $g'(x) = 12x^3 - 12x^2 - 72x = 12x(x-3)(x+2)$ and $g''(x) = 36x^2 - 24x - 72 = 12(3x^2 - 2x - 6)$.

You should discuss the domain, limits, critical points, intervals of increase and decrease, concavity, and possible points of inflection.

Solution:

(i) The domain is all reals.

(ii) The roots are at $x = -2, 4/3, 4$.

(iii) We have $\lim_{x\to+\infty} g(x) = \lim_{-\infty} g(x) = +\infty$.

(iv)
$$g'(x) = 12x^3 - 12x^2 - 72x = 12x(x^2 - x - 6) = 12x(x-3)(x+2).$$

This is defined everywhere, and has roots at $-2, 0, 3$.
Thus the critical points are $x = -2, 0, 3$. We compute $g(-2) = 0, g(0) = 64, g(3) = 5^2 \cdot 5 \cdot (-1) = -125$.

(v) We make a chart:

	$12x$	$(x-3)$	$(x+2)$	$g'(x)$
$x < -2$	$-$	$-$	$-$	$-$
$-2 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 3$	$+$	$-$	$+$	$-$
$3 < x$	$+$	$+$	$+$	$+$

This implies relative minima at -2 and at 3, and a relative maximum at 0.

(vi) We compute
$$g''(x) = 36x^2 - 24x - 72 = 12(3x^2 - 2x - 6)$$
which has roots
$$x = \frac{2 \pm \sqrt{4+72}}{6} = \frac{1 \pm \sqrt{19}}{3}.$$
and there are critical points at roughly -1 and $5/3$.
Plugging in some values we have

$$g''(2) = 12 \cdot (2) = 24 > 0$$
$$g''(0) = -72 < 0$$
$$g''(-2) = 12 \cdot 10 = 120 > 0.$$

Thus the function is concave up on $\left(-\infty, \frac{1-\sqrt{19}}{3}\right) \cup \left(\frac{1+\sqrt{19}}{3}, +\infty\right)$ and is concave down on $\left(\frac{1-\sqrt{19}}{3}, \frac{1+\sqrt{19}}{3}\right)$.

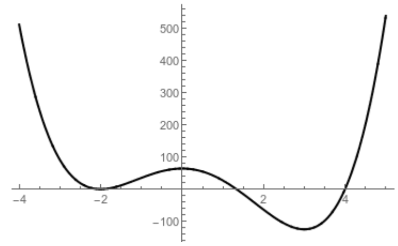

Figure 1: Graph of $g(x)$

14. **First and Second Derivative Tests**

(a) Classify all the critical points and relative extrema of $f(x) = \frac{x}{x^2+1}$. (For each critical point, tell me whether it is a relative maximum, a relative minimum, or neither.)

Solution:

We have

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2}$$

The critical points are thus at ± 1. We can make a chart:

	$(1-x)$	$1+x$	$(x^2+1)^{-2}$	$h'(x)$
$x < -1$	$+$	$-$	$+$	$-$
$-1 < x < 1$	$+$	$+$	$+$	$+$
$1 < x$	$-$	$+$	$+$	$-$

This tells us that we have a local minimum at $x = -1$ and a local maximum at $x = 1$.

Alternatively we could use the second derivative test. We have

$$f''(x) = \frac{-2x(x^2+1)^2 - 2(x^2+1)2x(1-x^2)}{(x^2+1)^4}$$
$$f''(-1) = \frac{8}{16} = 1/2 > 0$$
$$f''(1) = \frac{-8}{16} = -1/2 < 0$$

so we see that there's a local minimum at -1 and a local maximum at 1.

(b) Classify the critical points and relative extrema of $g(x) = \cos^2(x) - 2\sin(x)$ on $[0, 2\pi]$

Solution: We have

$$g'(x) = -2\cos(x)\sin(x) - 2\cos(x) = -2\cos(x)(\sin(x) + 1)$$

so $g'(x)$ is defined everywhere, and is 0 at $\pi/2, 3\pi/2$.

Here it looks easiest to use the second derivative test. We compute:

$$g''(x) = 2\sin^2(x) - 2\cos^2(x) + 2\sin(x)$$
$$g''(\pi/2) = 2 - 0 + 2 = 4 > 0$$
$$g''(3pi/2) = 2 - 0 - 2 = 0.$$

So this tells us that g has a local minimum at $\pi/2$, but doesn't tell us what happens at $3\pi/2$.

To answer that question we need the first derivative. If we make a chart we can plug in values like

$$g'(0) = -2$$
$$g'(\pi) = 2$$
$$g''(2\pi) = -2$$

	$g'(x)$
$0 \le x < \pi/2$	$-$
$\pi/2 < x < 3\pi/2$	$+$
$3\pi/2 < x \le 2\pi$	$-$

so g has a relative minimum at $\pi/2$ and a relative maximum at $3\pi/2$.

13. **Global Maxima and Critical Points**

(a) Find the absolute extrema of $f(x) = 3x^4 - 20x^3 + 24x^2 + 7$ on $[0, 5]$.

Solution: f is a continuous function on a closed ineterval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 12x^3 - 60x^2 + 48x = 12x(x^2 - 5x + 4) = 12x(x-4)(x-1)$ is defined everywhere and has roots at $0, 1, 4$. The endpoints are $0, 5$, so we need to evaluate f at $0, 1, 4, 5$.

$$f(0) = 7 \qquad f(1) = 14$$
$$f(4) = 3(4^4) - 5(4^4) + \frac{3}{2}(4^4) + 7 = \frac{-1}{2}4^4 + 7 = 7 - 128 = -121$$
$$f(5) = 3 \cdot 5^4 - 4 \cdot 5^4 + 5^4 - 5^2 + 7 = 7 - 25 = -18.$$

So the absolute maximum is 14 at 1, and the absolute minimum is -121 at 4.

(b) Find all the critical points of $g(x) = \dfrac{x^2 - 8}{x + 3}$.

Solution: The function is undefined at $x = -3$.
$g'(x) = \frac{2x(x+3) - 1(x^2-8)}{(x+3)^2} = \frac{x^2+6x+8}{(x+3)^2}$. The denominator is zero when $x = -3$, and thus the derivative is undefined there, but so is the function. The numerator is $(x + 2)(x + 4)$ and thus has roots when $x = -2, -4$. So the critical points of the function are $-2, -3, -4$.

9. **Linear Approximation and Tangent Lines**

(a) Estimate $\sqrt[4]{15}$ using a linear approximation of the function $\sqrt[4]{x}$ at the point 16.

Solution: We have $h(x) = \sqrt[4]{x}$ and so $h'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4\sqrt[4]{x^3}}$. Thus in particular, we have $h'(16) = \frac{1}{4\sqrt[4]{16^3}} = \frac{1}{4 \cdot 2^3} = 1/32$.

The tangent line approximation is

$$y - 2 = \frac{1}{32}(x - 16)$$

so we have

$$
\begin{aligned}
f(x) &\approx \frac{1}{32}(x-16)+2 \\
f(16) &\approx \frac{1}{32}(-1)+2 = 2-\frac{1}{32} = \frac{63}{32}.
\end{aligned}
$$

(b) Find an equation of the line tangent to $y = \frac{x+1}{x-1}$ at the point $x = 2$.

Solution:

$$y' = \frac{(x-1)-(x+1)}{(x-1)^2}$$

and thus at $x = 2$ we have $y' = \frac{1-3}{1^2} = -2$. The point on the curve is $(2, 3)$, so we have the equation

$$y - 3 = -2(x-2) \qquad \text{or} \qquad y = 7 - 2x.$$