Math 1231 Fall 2020
Single-Variable Calculus I Mastery Quiz 12
Due midnight on Thursday, December 3

This week's mastery quiz has seven topics. **Do not answer all five.** You may answer the two newest topics, numbered 20 and 19, and *one* additional topic. You may pick one topic you have not yet demonstrated mastery on and answer the questions on that topic. (If you are retrying a topic, please complete the entire page.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 10-20 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics:

20. The Evaluation Theorem and Definite Integrals
19. The Fundamental Theorem of Calculus and Antiderivatives
18. Area and Riemann Sums
17. Approximation
16. Optimization
13. Global Maxima and Critical Points
2. Formal limits

20. **The Evaluation Theorem and Definite Integrals**

Compute:

(a) $\int_{-2}^{4} x^3 - 3x\,dx =$ **Solution:**

$$\begin{aligned}\int_{-2}^{4} x^3 - 3x\,dx &= \frac{x^4}{4} - \frac{3x^2}{2}\Big|_{-2}^{4} \\ &= 64 - 4 - 24 + 6 = 42.\end{aligned}$$

(b) $\int_{\pi/6}^{3\pi/4} \csc^2(t)\,dt =$

Solution:

$$\begin{aligned}\int_{\pi/6}^{3\pi/4} \csc^2(t)\,dt &= -\cot(t)\Big|_{\pi/6}^{3\pi/4} \\ &= \frac{-\cos(3\pi/4)}{\sin(3\pi/4)} - \frac{-\cos(\pi/6)}{\sin(\pi/6)} \\ &= \frac{\sqrt{2}/2}{\sqrt{2}/2} + \frac{\sqrt{3}/2}{1/2} = 1 + \sqrt{3}.\end{aligned}$$

19. **The Fundamental Theorem of Calculus and Antiderivatives**

(a) Let $F(x) = \int_{2}^{\sqrt{x^2+1}} t\sin(t)\,dt$. What is $F'(x)$?

Solution: If we set $F_1(x) = \int_2^x t\sin(t)\,dt$ then $F_1'(x) = x\sin(x)$, so

$$\frac{d}{dx}F(x) = \frac{d}{dx}F_1(\sqrt{x^2+1}) = \sqrt{x^2+1}\sin(\sqrt{x^2+1})\frac{2x}{2\sqrt{x^2+1}}.$$

(b) Find an antiderivative of $g(x) = \cos(x) + 2x$.

Solution: $\sin(x) + x^2$.

(c) Compute $\int x^2\sqrt{x}\,dx$.

Solution:

$\int x^2\sqrt{x}\,dx = \int x^{5/2}\,dx = \frac{2}{7}x^{7/2} + C$.

18. **Area and Riemann Sums**

Let $f(x) = 2x^3$ be defined on the interval $[0, 4]$.

(a) Approximate the area under the curve of the function using four rectangles and right endpoints.

(b) Approximate the area under the curve of the function using four rectangles and left endpoints.

(c) Find a formula for computing R_n, the estimate using n rectangles and right endpoints. (This formula should not have a summation sign or be given as a sum of n terms.)

(d) Use the formula in part (c) to compute the area exactly.

Solution:

(a) $R_4 = 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) = 2 + 16 + 54 + 128 = 200$

(b) $L_4 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) = 0 + 2 + 16 + 54 = 72$.

(c)

$$\begin{aligned} R_n &= \sum_{i=1}^{n} \frac{4}{n} f\left(0 + i\frac{4}{n}\right) = \sum_{i=1}^{n} \frac{4}{n} 2\left((4i/n)\right)^3 \\ &= \sum_{i=1}^{n} \frac{8}{n} \frac{64i^3}{n^3} \\ &= \sum_{i=1}^{n} \frac{512i^3}{n^4} \\ &= \frac{512}{n^4} \sum_{i=1}^{n} i^3 \\ &= \frac{512}{n^4} \frac{n^2(n+1)^2}{4} \\ &= \frac{128n^2(n+1)^2}{n^4} \end{aligned}$$

(d) We can compute

$$\lim_{n \to +\infty} R_n = \lim_{n \to +\infty} \frac{128n^2(n+1)^2}{n^4} = 128.$$

17. **Approximation**

(a) Find a formula for the quadratic approximation of $f(x) = \sin(x^2 + x)$ near the point $a = 0$, and use it to estimate $f(.1)$.

Solution: We compute

$$\begin{aligned} & & f(0) &= 0 \\ f'(x) &= \cos(x^2 + x)(2x + 1) & f'(0) &= 1 \\ f''(x) &= -\sin(x^2 + x)(2x + 1)^2 + 2\cos(x^2 + x) & f''(0) &= 2 \end{aligned}$$

and thus we have

$$\begin{aligned} f(x) &\approx 0 + 1(x) + \frac{2}{2}(x)^2 = x + x^2 \\ f(.1) &\approx .1 + .01 = .11. \end{aligned}$$

(b) Use two steps of Newton's method to estimate a solution to $x^3 + x = 1$ starting from $x_0 = 1$. (You should compute x_2.)

Solution: We're looking for a root of $g(x) = x^3 + x - 1$. So we compute $g'(x) = 3x^2 + 1$, and we get

$$x_1 = 1 - \frac{g(1)}{g'(1)} = 1 - \frac{1}{4} = \frac{3}{4}$$
$$x_2 = \frac{3}{4} - \frac{g(3/4)}{g'(3/4)} = \frac{3}{4} - \frac{27/64 + 3/4 - 1}{27/16 + 1} = \frac{3}{4} - \frac{11/64}{43/16} = \frac{3}{4} - \frac{11}{172} = \frac{118}{172} \approx .686.$$

(The true answer is approximately .68233, so yay for us.)

16. **Optimization**

A poster needs to have an area of $216in^2$, with 1-inch margins on the bottom and sides and a 2-inch margin on the top. What dimension maximize the area of the *printed* region, excluding margins? **Solution:** Our constraint is that $h \cdot w = 216$, and we want to maximize $(h-3)(w-2)$. So we have $h = 216/w$, and thus we want to maximize

$$A(w) = \left(\frac{216}{w} - 3\right)(w - 2) = 180 - 3w - \frac{432}{w} + 6$$
$$A'(w) = -3 + \frac{432}{w^2}$$
$$w^2 = 144$$
$$w = \pm 12.$$

Obviously the only relevant critical point is the positive one, 12.

We can check that this is truly a maximum by the second derivative: $A''(w) = \frac{-720}{w^3} < 0$ for any $w > 0$, so we have a local maximum.

Or we can see that $A'(x) > 0$ when w is small and $A'(w) < 0$ when w is large, so the function is increasing until 80 and decreasing after.

So the dimensions we want are a poster with width 12 inches and height $216/12 = 18$ inches.

13. **Global Maxima and Critical Points**

(a) Find the absolute extrema of $f(x) = 3x^4 - 20x^3 + 24x^2 + 7$ on $[0, 5]$, and justify your claim that they are absolute extrema.

Solution: f is a continuous function on a closed interval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 12x^3 - 60x^2 + 48x = 12x(x^2 - 5x + 4) = 12x(x-4)(x-1)$ is defined everywhere and has roots at $0, 1, 4$. The endpoints are $0, 5$, so we need to evaluate f at $0, 1, 4, 5$.

$$f(0) = 7 \qquad f(1) = 14$$
$$f(4) = 3(4^4) - 5(4^4) + \frac{3}{2}(4^4) + 7 = \frac{-1}{2}4^4 + 7 = 7 - 128 = -121$$
$$f(5) = 3 \cdot 5^4 - 4 \cdot 5^4 + 5^4 - 5^2 + 7 = 7 - 25 = -18.$$

So the absolute maximum is 14 at 1, and the absolute minimum is -121 at 4.

(b) Find all the critical points of $g(x) = \dfrac{x^3 + 3x^2 - 4}{x^2}$.

Solution:

$$\begin{aligned} g'(x) &= \frac{(3x^2 + 6x)(x^2) - 2x(x^3 + 3x^2 - 4)}{x^4} \\ &= \frac{3x^4 + 6x^3 - 2x^4 - 6x^3 + 8x}{x^4} \\ &= \frac{x^4 + 8x}{x^4} = \frac{x^3 + 8}{x^3} \end{aligned}$$

and so $g'(x) = 0$ when $x = -\sqrt[3]{8}$ and $g'(x)$ is undefined when $x = 0$. Thus the critical points are $-2, 0$.

2. **Formal Limits**

(a) Write a formal ϵ-δ proof that $\lim_{x \to 1} 3 - x = 2$.

Solution: Let $\epsilon > 0$ and set $\delta = \epsilon$. Then if $0 < |x - 1| < \delta$, we have

$$|3 - x - 2| = |1 - x| = |x - 1| < \delta = \epsilon.$$

(b) Explicitly naming each limit law you use, compute

$\lim\limits_{x \to 2} \sqrt{(x+2)(x-1) + 5} =$

Solution:

$$\begin{aligned} \lim_{x \to 2} \sqrt{(x+2)(x-1)+5} &= \sqrt{\lim_{x \to 2} (x+2)(x-1) + 5} && \text{Exponents} \\ &= \sqrt{\lim_{x \to 2} (x+2)(x-1) + \lim_{x \to 2} 5} && \text{Sums} \\ &= \sqrt{\lim_{x \to 2} (x+2) \lim_{x \to 2} (x-1) + \lim_{x \to 2} 5} && \text{Products} \\ &= \sqrt{(\lim_{x \to 2} x + \lim_{x \to 2} 2)(\lim_{x \to 2} x - \lim_{x \to 2} 1) + \lim_{x \to 2} 5} && \text{Sums} \\ &= \sqrt{(\lim_{x \to 2} x + 2)(\lim_{x \to 2} x - 1) + 5} && \text{constants} \\ &= \sqrt{(2+2)(2-1)+5} = 3 && \text{Identity} \end{aligned}$$