Math 1231 Fall 2020
Single-Variable Calculus I Mastery Quiz 13
Due midnight on Thursday, December 10

This week's mastery quiz has eight topics. **Do not answer all eight.** You may answer the two newest topics, numbered 22 and 21, and *one* additional topic. You may pick one topic you have not yet demonstrated mastery on and answer the questions on that topic. (If you are retrying a topic, please complete the entire page.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics:

22. Areas and Averages
21. Integrals by Substitution
20. The Evaluation Theorem and Definite Integrals
19. The Fundamental Theorem of Calculus and Antiderivatives
18. Area and Riemann Sums
17. Approximation
12. Related Rates
4. Trigonometric Limits

22. **Areas and Averages**

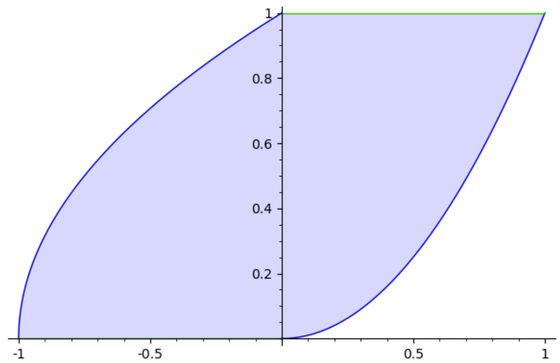
(a) What is the average value of the function $f(x) = x^3 - x$ on the interval $[0, 2]$?

Solution:

$$\frac{1}{2}\int_0^2 x^3 - x\,dx = \frac{1}{2}\frac{x^4}{4} - \frac{x^2}{2}\Big|_0^2 = \frac{1}{2}(4 - 2 - (0 - 0)) = 1.$$

(b) Sketch the region bounded by the curves $x = y^2 - 1$, $y = 0$, $y = 1$, and $x = \sqrt{y}$, and find the area of that region.

Solution:



We really want to integrate this with respect to y. So we have

$$\int_0^1 \sqrt{y} - (y^2 - 1)\,dy = \int 1 + \sqrt{y} - y^2\,dy$$
$$= y + \frac{2}{3}y^{3/2} - \frac{y^3}{3}\Big|_0^1 = 1 + \frac{2}{3} - \frac{1}{3} = \frac{4}{3}.$$

If we really want to, though, we can integrate with respect to x. We compute

$$A = \int_{-1}^0 \sqrt{x+1}\,dx + \int_0^1 1 - x^2\,dx$$
$$= \frac{2}{3}(x+1)^{3/2}\Big|_{-1}^0 + x - \frac{x^3}{3}\Big|_0^1$$
$$= \frac{2}{3} - 0 + 1 - \frac{1}{3} - 0 + 0 = \frac{4}{3}.$$

21. **Integrals by Substitution**

(a) Compute $\int \cos(5x + 3)\,dx =$

Solution: Set $u = 5x + 3$ so $du = 5dx$ and $dx = du/5$. Then

$$\int \cos(5x + 3)\,dx = \int \cos(u)\frac{du}{5} = \frac{\sin(u)}{5} + C.$$

(b) Compute $\int x^2\sqrt{x+3}\,dx =$

Solution: We take $u = x+3$ so $du = dx$ and also $x = u-3$. Then we have

$$\begin{aligned}\int x^2\sqrt{x+3}\,dx &= \int (u-3)^2\sqrt{u}\,du = \int (u^2-6u+9)\sqrt{u}\,du \\ &= \int u^{5/2} - 6u^{3/2} + 9u^{1/2}\,du = \frac{2}{7}u^{7/2} - \frac{12}{5}u^{5/2} + 6u^{3/2} + C \\ &= \frac{2}{7}(x+3)^{7/2} - \frac{12}{5}(x+3)^{5/2} + 6(x+3)^{3/2} + C.\end{aligned}$$

(c) **By changing the bounds of the integral** compute $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx =$

Solution: Take $u = x^2$ so $du = 2x\,dx$ and $dx = du/2x$. We then have $g(0) = 0^2 = 0$ and $g(\sqrt{\pi}) = \sqrt{\pi}^2 = \pi$ so we get

$$\begin{aligned}\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx &= \int_0^{\pi} x\sin(u)\frac{du}{2x} \\ &= \frac{1}{2}\int_0^{\pi} -\cos(u)\,du = \frac{-1}{2}\cos(u)\Big|_0^{\pi} = \frac{-1}{2}(-1-1) = 1.\end{aligned}$$

20. **The Evaluation Theorem and Definite Integrals**

Compute:

(a) $\int_{-1}^{2} 3x^2 - x\,dx =$ **Solution:**

$$\begin{aligned}\int_{-1}^{2} 3x^2 - x\,dx &= x^3 - \frac{x^2}{2}\Big|_{-1}^{2} \\ &= 8 - 2 - (-1 - 1/2) = 15/2.\end{aligned}$$

(b) $\int_0^{\pi/3} \sec(t)\tan(t)\,dt =$

Solution:

$$\begin{aligned}\int_0^{\pi/3} \sec(t)\tan(t)\,dt &= \sec(t)\Big|_0^{\pi/3} \\ &= \frac{1}{\cos(\pi/3)} - \frac{1}{\cos(0)} \\ &= 2 - 1 = 1.\end{aligned}$$

19. **The Fundamental Theorem of Calculus and Antiderivatives**

(a) Let $F(x) = \int_1^{3x^2} e^{\sec(t)}\,dt$. What is $F'(x)$?

Solution: If we set $F_1(x) = \int_2^x e^{\sec(t)}\,dt$ then $F_1'(x) = e^{\sec(x)}$, so

$$\frac{d}{dx}F(x) = \frac{d}{dx}F_1(3x^2) = F_1'(3x^2)\cdot 6x = e^{\sec(3x^2)}6x.$$

(b) Find an antiderivative of $g(x) = 4x^4$.
Solution: $\frac{4}{5}x^5$.

(c) Compute $\int \sqrt{x} + \sqrt[3]{x}\,dx$.
Solution:
$\int \sqrt{x} + \sqrt[3]{x}\,dx = \frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C$.

18. **Area and Riemann Sums**

Let $f(x) = 2x + 4x^2$ be defined on the interval $[0, 2]$.

(a) Approximate the area under the curve of the function using four rectangles and right endpoints.

(b) Approximate the area under the curve of the function using four rectangles and left endpoints.

(c) Find a formula for computing R_n, the estimate using n rectangles and right endpoints. (This formula should not have a summation sign or be given as a sum of n terms.)

(d) Use the formula in part (c) to compute the area exactly.

Solution:

(a) $R_4 = 1/2 \cdot f(1/2) + 1/2 \cdot f(1) + 1/2 \cdot f(3/2) + 1/2 \cdot f(2) = \frac{1}{2}(2 + 6 + 12 + 20) = 20$

(b) $L_4 = 1/2 \cdot f(0) + 1/2 \cdot f(1/2) + 1/2 \cdot f(1) + 1/2 \cdot f(3/2) = \frac{1}{2}(0 + 2 + 6 + 12) = 10$

(c)

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} \frac{2}{n} f\left(0 + i\frac{2}{n}\right) = \sum_{i=1}^{n} \frac{2}{n}\left(2(2i/n) + 4(2i/n)^2\right) \\
&= \sum_{i=1}^{n} \frac{2}{n}\left(\frac{4i}{n} + \frac{16i^2}{n^2}\right) \\
&= \sum_{i=1}^{n} \frac{8i}{n^2} + \frac{32i^2}{n^3} \\
&= \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{8}{n^2}\frac{n(n+1)}{2} + \frac{32}{n^3}\frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

(d) We can compute

$$\begin{aligned}
\lim_{n\to+\infty} R_n &= \lim_{n\to+\infty} \frac{8}{n^2}\frac{n(n+1)}{2} + \frac{32}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= 4 + \frac{32}{3} = \frac{44}{3}.
\end{aligned}$$

17. **Approximation**

(a) Find a formula for the quadratic approximation of $f(x) = \frac{2x}{x+2}$ near the point $a = 2$, and use it to estimate $f(1.9)$.

Solution: We compute

$$f(1) = 1$$

$$f'(x) = \frac{2(x+2) - 2x}{(x+2)^2} = \frac{4}{(x+2)^2} \qquad f'(2) = 1/4$$

$$f''(x) = \frac{-8}{(x+2)^3} \qquad f''(2) = \frac{-1}{8}$$

and thus we have

$$\begin{aligned} f(x) &\approx 1 + \frac{1}{4}(x-2) - \frac{1}{16}(x-2)^2 \\ f(1.9) &\approx 1 - \frac{1}{4}(.1) - \frac{1}{16}(.1)^2 \\ &= \frac{1159}{1600} = 1 - .025 - .000625 = .974375 \end{aligned}$$

(b) Use two steps of Newton's method to estimate a solution to $\frac{2}{x} + 1 = x^2$ starting from $x_0 = 1$. (You should compute x_2.)

Solution: We're looking for a root of $g(x) = \frac{2}{x} - x^2 + 1$. So we compute $g'(x) = -\frac{2}{x^2} - 2x$, and we get

$$x_1 = 1 - \frac{2}{-4} == \frac{3}{2}$$

$$x_2 = \frac{3}{2} - \frac{g(3/2)}{g'(3/2)} = \frac{3}{2} - \frac{4/3 - 9/4 + 1}{-8/9 - 3} = \frac{3}{2} - \frac{1/12}{-35/9} = \frac{3}{2} + \frac{3}{140} = \frac{213}{140} \approx 1.5214285.$$

(The true answer is approximately 1.5213797, so yay for us.)

12. **Related Rates**

A snowball is melting such that its surface area is decreasing at $1\text{cm}^2/\text{min}$. When the radius is 8cm, how quickly is the radius decreasing? Please **write a complete sentence** to answer this question at the end of your work.

(The surface area of a sphere of radius r is $4\pi r^2$.)

Solution: We have $S = 4\pi r^2$, so $S' = 8\pi r r'$. When the radius is 8cm we have

$$\begin{aligned} S' &= 8\pi \cdot 8\text{cm} \cdot r' \\ -1\text{cm}^2/\text{min} &= 64\pi r' \\ r' &= \frac{-1}{64\pi}\text{cm/min}. \end{aligned}$$

Thus when the radius is 8cm, the radius of the snowball is decreasing by $\frac{1}{64\pi}$ centimeters per minute.

4. **Trigonometric Limits**

(a) Show that $\lim_{x\to -2}(x+2)\sin\left(\frac{3x}{x+2}\right) = 0.$

Solution: We know that

$$-1 \le \sin\left(\frac{3x}{x+2}\right) \le 1$$
$$-|x+2| \le (x+2)\sin\left(\frac{3x}{x+2}\right) \le |x+2|$$

Since $\lim_{x\to -2} x+2 = 0$ and $\lim_{x\to -2} x+2 = 0$, by the Squeeze Theorem, we know that $\lim_{x\to -2}(x+2)\sin\left(\frac{3x}{x+2}\right) = 0$.

(b) Compute $\lim_{x\to 0}\frac{x\sin(4x)}{\sin(x)\sin(2x)}$

Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{x\sin(4x)}{\sin(x)\sin(2x)} &= \lim_{x\to 0}\frac{x}{\sin x}\frac{\sin(4x)}{4x}\frac{2x}{\sin(2x)}\cdot 2 \\ &= 1\cdot 1\cdot 1\cdot 2 = 2.\end{aligned}$$