Math 1231 Fall 2020
Single-Variable Calculus I Mastery Quiz 3
Due midnight on Thursday, September 24

This week's mastery quiz has five topics. **Do not answer all five.** You may answer the two first questions on the newest topics, numbered 5 and 4, and *one* additional topic of the previous three. Pick one topic you have not yet demonstrated mastery on and answer the question on that topic. (If you have already demonstrated mastery on topics 1-3, you don't need to answer any of them.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 10-20 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics:

5. Infinite Limits
4. Trigonometric Limits
3. Computing Limits
2. Formal limits
1. Informal limits and continuity

5. **Infinite Limits** Compute:

(a) $\lim_{x\to -2} \dfrac{x-2}{(x+2)^2} =$

Solution: The top approaches -4 and the bottom approaches 0, so

$$\lim_{x\to -2} \frac{x-2}{(x+2)^2} = \pm\infty.$$

Further, we see that the top is negative and the bottom is always positive, so in fact the limit is $-\infty$.

(b) $\lim_{x\to 3} \dfrac{5x}{x-3} =$

Solution: The limit of the top is 15 and the limit of the bottom is 0, so

$$\lim_{x\to 3} \frac{5x}{x-3} = \pm\infty.$$

Since the denominator can be either positive or negative, we can't be more precise than this.

(c) $\lim_{x\to -\infty} \dfrac{3x^2+2x+1}{\sqrt{x^4-x^2+x}} =$

Solution:

$$\begin{aligned}
\lim_{x\to -\infty} \frac{3x^2+2x+1}{\sqrt{x^4-x^2+x}} &= \lim_{x\to -\infty} \frac{3+2/x+1/x^2}{1/\sqrt{x^4}\sqrt{x^4-x^2+x}} \\
&= \lim_{x\to -\infty} \frac{3+2/x+1/x^2}{\sqrt{1-1/x^2+1/x^3}} \\
&= \frac{3+0+0}{\sqrt{1-0+0}} = 3.
\end{aligned}$$

4. **Trigonometric Limits**

(a) Show that $\lim_{x\to 2} (x-2)\left(1+\sin\left(\dfrac{1}{x-2}\right)\right) = 0.$

Solution: We know that

$$\begin{aligned}
-1 \le \sin\left(\frac{1}{x-2}\right) &\le 1 \\
0 \le 1+\sin\left(\frac{1}{x-2}\right) &\le 2 \\
0 \le (x-2)\left(1+\sin\left(\frac{1}{x-2}\right)\right) &\le 2|x-2|.
\end{aligned}$$

Since $\lim_{x\to 2} 0 = 0$ and $\lim_{x\to 2} 2|x-2| = 0$, by the Squeeze Theorem, we know that $\lim_{x\to 2}(x-2)\left(1+\sin\left(\frac{1}{x-2}\right)\right) = 0$.

3. **Computing Limits** Compute:

(a) $\lim_{x\to 1} \dfrac{x^2+3x-4}{x-1} =$
Solution:

$$\lim_{x\to 1} \frac{x^2+3x-4}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+4)}{x-1} = \lim_{x\to 1} x+4 = 5$$

(b) $\lim_{x\to -2} \dfrac{1}{2x+4} + \dfrac{2}{2x^2+4x} =$
Solution:

$$\lim_{x\to -2} \frac{1}{2x+4} + \frac{1}{2x+x^2} = \lim_{x\to -2} \frac{2x+x^2+2x+4}{(2x+4)(2x+x^2)} = \lim_{x\to -2} \frac{x^2+4x+4}{2x(x+2)^2} = \lim_{x\to -2} \frac{1}{2x} = \frac{-1}{4}.$$

(c) $\lim_{x\to 1} \dfrac{\sqrt{x+8}-3}{x-1} =$
Solution:

$$\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1} = \lim_{x\to 1} \frac{x+8-9}{(x-1)(\sqrt{x+8}+3)} = \lim_{x\to 1} \frac{1}{\sqrt{x+8}+3} = \frac{1}{6}.$$

2. **Formal Limits**

(a) Write a formal ϵ-δ proof that $\lim_{x\to 1} 4x-2 = 2$.
Solution: Let $\epsilon > 0$ and set $\delta = \epsilon/4$. Then if $0 < |x-1| < \delta$, we have

$$|4x-2-2| = |4x-4| = 4|x-1| < 4\delta = \epsilon.$$

(b) Explicitly naming each limit law you use, compute
$\lim_{x\to -1} \dfrac{x^3+x}{3(x-2)} =$
Solution:

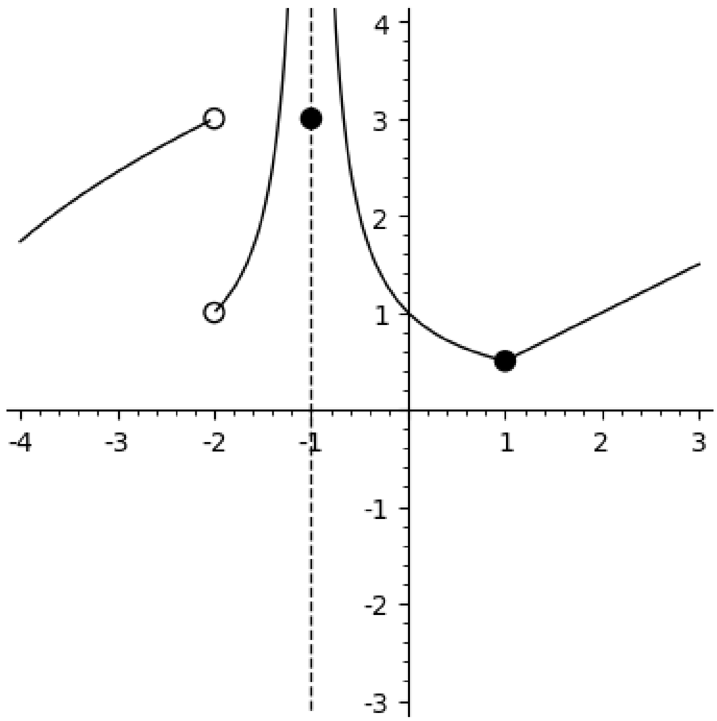
$$\begin{aligned}
\lim_{x\to -1} \frac{x^3+x}{3(x-2)} &= \frac{\lim_{x\to -1} x^3+x}{\lim_{x\to -1} 3(x-2)} && \text{Quotients}\\
&= \frac{\lim_{x\to -1} x^3+x}{\lim_{x\to -1} 3 \cdot \lim_{x\to -1} x-2} && \text{Products}\\
&= \frac{\lim_{x\to -1} x^3 + \lim_{x\to -1} x}{\lim_{x\to -1} 3 \cdot (\lim_{x\to -1} x - \lim_{x\to -1} 2)} && \text{additivity}\\
&= \frac{(\lim_{x\to -1} x)^3 + \lim_{x\to -1} x}{\lim_{x\to -1} 3 \cdot (\lim_{x\to -1} x - \lim_{x\to -1} 2)} && \text{exponents}\\
&= \frac{(-1)^3+(-1)}{\lim_{x\to -1} 3\,(-1 - \lim_{x\to -1} 2)} && \text{identity}\\
&= \frac{-1-1}{3(-1-2)} = \frac{2}{9} && \text{constants.}
\end{aligned}$$

1. **Informal Continuity and Limits**

(a) Give a (zeroth-order) approximate value for $\sqrt[3]{-9}$, and explain how you got it.

Solution: We know that -9 is close to -8, so we expect $\sqrt[3]{-9}$ to be close to $\sqrt[3]{-8} = -2$.

Here is the graph of a function f:



For each of the following questions, if your answer is "does not exist", explain in a few words why it does not exist. If your answer is just a number, you don't need to explain.

(b) What is the domain of f?

Solution: All reals except -2. I'd also except all of $[-4, 3]$ except -2.

(c) Where (if anywhere) is f discontinuous?

Solution: $x = -2, -1$.

(d) What is $\lim_{x \to -2} f(x)$?

Solution: Does not exist, since the two one-sided limits don't agree.

(e) What is $\lim_{x \to -2^+} f(-2)$?

Solution: 1

(f) What is $\lim_{x \to -1} f(x)$?

Solution: $+\infty$. I think I'm okay with people just saying the limit doesn't exist in this context, though.

(g) What is $f(-1)$?

Solution: 3