Math 1231 Fall 2020
Single-Variable Calculus I Mastery Quiz 4
Due midnight on Thursday, October 1

This week's mastery quiz has five topics. **Do not answer all five.** You may answer the two first questions on the newest topics, numbered 7 and 6, and *one* additional topic of the previous three. You may pick one topic you have not yet demonstrated mastery on and answer the question on that topic. (If you are retrying a topic, please complete the entire page.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 10-20 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics:

7. Basics of Computing Derivatives
6. Definition of a Derivative
5. Infinite Limits
4. Trigonometric Limits
3. Computing Limits
2. Formal limits
1. Informal limits and continuity

7. **Basics of Computing Derivatives**

(a) Compute the derivative of $f(x) = (x+2)\sqrt[3]{x} + x^3$ while explicitly naming every derivative rule you use.

Solution:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(x+2)\sqrt[3]{x} + \frac{d}{dx}x^3 && \text{Sum Rule}\\
&= (x+2)'\sqrt[3]{x} + (x+2)\frac{d}{dx}\sqrt[3]{x} + \frac{d}{dx}x^3 && \text{Product Rule}\\
&= (x+2)'\sqrt[3]{x} + (x+2)\frac{1}{3}x^{-2/3} + 3x^2 && \text{Power Rule}\\
&= (x'+2')\sqrt[3]{x} + (x+2)\frac{1}{3}x^{-2/3} + 3x^2 && \text{Sum Rule}\\
&= (1+2')\sqrt[3]{x} + (x+2)\frac{1}{3}x^{-2/3} + 3x^2 && \text{Identity (or Power Rule)}\\
&(1+0)\sqrt[3]{x} + (x+2)\frac{1}{3}x^{-2/3} + 3x^2 && \text{Constants.}
\end{aligned}$$

Compute the derivative each of the following functions, using any tools we have developed in class.

(b) $\dfrac{3x^2-1}{\sqrt{x}}$

Solution:

$$\frac{6x\sqrt{x} - \frac{1}{2}x^{-1/2}(3x^2-1)}{x}$$

(c) $(5x^7 - 3x)\left(x^{4/3} + \dfrac{1}{x}\right)$

Solution:

$$(35x^6 - 3)\left(x^{4/3} + \frac{1}{x}\right) + \left(\frac{4}{3}x^{1/3} - \frac{1}{x^2}\right)(5x^7 - 3x).$$

6. **Definition of a Derivative**

Compute the following derivatives, *directly from the formal definition of derivative.*

(a) If $f(x) = x^2 + 2x$, find $f'(2)$. **Solution:**

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}\\
&= \lim_{h\to 0} \frac{(2+h)^2 + 2(2+h) - 8}{h}\\
&= \lim_{h\to 0} \frac{4 + 4h + h^2 + 4 + 2h - 8}{h}\\
&= \lim_{h\to 0} \frac{h^2 + 6h}{h}\\
&= \lim_{h\to 0} h + 6 = 6.
\end{aligned}$$

(b) If $g(x) = \frac{1}{x+2}$, find $g'(x)$. **Solution:**

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} \\ &= \lim_{h\to 0} \frac{(x+2)-(x+h+2)}{(x+2)(x+h+2)h} \\ &= \lim_{h\to 0} \frac{-h}{(x+2)(x+h+2)h} \\ &= \lim_{h\to 0} \frac{-1}{(x+2)(x+h+2)} = \frac{-1}{(x+2)^2}. \end{aligned}$$

5. **Infinite Limits** Compute:

(a) $\displaystyle\lim_{x\to-\infty} \frac{x^3+x+1}{\sqrt{x^6-3x^3}}$ **Solution:**

$$\begin{aligned} \lim_{x\to-\infty} \frac{x^3+x+1}{\sqrt{x^6-3x^3}} &= \lim_{x\to-\infty} \frac{1+1/x^2+1/x^3}{\frac{-1}{\sqrt{x^6}}\sqrt{x^6-3x^3}} \\ &= \lim_{x\to-\infty} \frac{1+1/x^2+1/x^3}{-\sqrt{1-3/x^3}} \\ &= \frac{1}{-1} = -1. \end{aligned}$$

(b) $\displaystyle\lim_{x\to 3} \frac{1-x}{(x-3)^3} =$

Solution: The top approaches -2 and the bottom approaches 0, so

$$\lim_{x\to 3} \frac{1-x}{(x-3)^3} = \pm\infty.$$

Since the denominator can be either positive or negative, we can't be any more precise than this.

(c) $\displaystyle\lim_{x\to 2} \frac{2x+1}{(x-2)^2} =$

Solution: The limit of the top is 5 and the limit of the bottom is 0, so

$$\lim_{x\to 2} \frac{2x+1}{(x-2)^2} = \pm\infty.$$

The numerator is positive and the denominator is always positive, so in fact the limit is $+\infty$.

4. **Trigonometric Limits**

(a) Show that $\displaystyle\lim_{x\to-1} (x+1)\sin\left(\frac{3}{(x+1)^3}\right) = 0.$

Solution: We know that

$$-1 \le \sin\left(\frac{3}{(x+1)^3}\right) \le 1$$
$$-|x+1| \le (x+1)\sin\left(\frac{3}{(x+1)^3}\right) \le |x+1|.$$

Since $\lim_{x\to -1} -|x+1| = 0$ and $\lim_{x\to -1} |x+1| = 0$, by the Squeeze Theorem, we know that $\lim_{x\to -1}(x+1)\sin\left(\frac{3}{(x+1)^3}\right) = 0$.

(b) Compute $\lim_{x\to 0} \frac{\sin(3x)\sin(4x)}{x\sin(2x)} =$

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(3x)\sin(4x)}{x\sin(2x)} &= \lim_{x\to 0} \frac{\sin(3x)}{3x}\frac{\sin(4x)}{4x}\frac{12x^2}{x\sin(2x)} \\ &= \lim_{x\to 0} \frac{2x}{\sin(2x)}\frac{6x}{x} \\ &= \lim_{x\to 0} \frac{6x}{x} = 6.\end{aligned}$$

3. **Computing Limits** Compute:

(a) $\lim_{x\to 2} \frac{x^2-4x+3}{x^2-3x+4} =$

Solution:

$$\lim_{x\to 2} \frac{x^2-4x+3}{x^2-3x+4} = \frac{-1}{2}.$$

(b) $\lim_{x\to 3} \frac{\sqrt{7-x}-2}{x-3} =$

Solution:

$$\lim_{x\to 3} \frac{\sqrt{7-x}-2}{x-3} = \lim_{x\to 3} \frac{7-x-4}{(x-3)(\sqrt{7-x}+2)} = \lim_{x\to 3} \frac{-1}{\sqrt{7-x}+2} = \frac{-1}{4}.$$

(c) $\lim_{x\to -1} \frac{x^2+6x+5}{x+1} =$

Solution:

$$\lim_{x\to -1} \frac{x^2+6x+5}{x+1} = \lim_{x\to -1} \frac{(x+1)(x+5)}{x+1} = \lim_{x\to 1} x+5 = 6.$$

2. **Formal Limits**

(a) Write a formal ϵ-δ proof that $\lim_{x\to 3} x+5 = 8$.

Solution: Let $\epsilon > 0$ and set $\delta = \epsilon$. Then if $0 < |x-3| < \delta$, we have

$$|x+5-8| = |x-3| < \delta = \epsilon.$$

(b) Explicitly naming each limit law you use, compute

$\lim_{x\to 0} \frac{x^2 - x}{2x} =$

Solution:

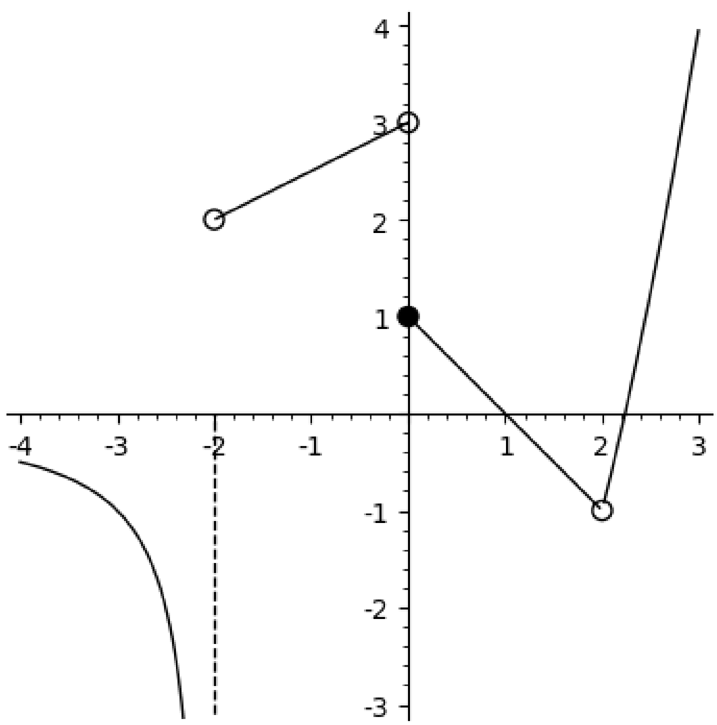
$$\begin{aligned}
\lim_{x\to 0} \frac{x^2 - x}{2x} &= \lim_{x\to 0} \frac{x-1}{2} && \text{Almost Identical Functions} \\
&= \frac{\lim_{x\to 0} x - 1}{\lim_{x\to 0} 2} && \text{Quotients} \\
&= \frac{\lim_{x\to 0} x - \lim_{x\to 0} 1}{\lim_{x\to 0} 2} && \text{Additivity} \\
&= \frac{\lim_{x\to 0} x - 1}{2} && \text{Constants} \\
&= \frac{-1}{2} && \text{Identity.}
\end{aligned}$$

1. **Informal Continuity and Limits**

(a) Give a (zeroth-order) approximate value for sin(3.14), and explain how you got it. (Note: if you think you can compute this exactly in your head, you are wrong.)

Solution: We know that 3.14 is close to π, so we expect sin(3.14) to be close to $\sin(\pi) = 0$.

Here is the graph of a function f:



For each of the following questions, if your answer is "does not exist", explain in a few words why it does not exist. If your answer is just a number, you don't need to explain.

(b) What is the domain of f?

Solution: All reals except $-2, 2$. I'd also except all of $[-4, 3]$ except $-2, 2$.

(c) Where (if anywhere) is f discontinuous?

Solution: $x = -2, 0, 2$.

(d) What is $\lim_{x \to 0} f(x)$?

Solution: Does not exist, since the two one-sided limits don't agree.

(e) What is $f(0)$?

Solution: 1

(f) What is $\lim_{x \to 2} f(x)$?

Solution: This limit is -1.

(g) What is $f(2)$?

Solution: Does not exist, since the function has no value there.