Math 1231 Fall 2020
Single-Variable Calculus I Mastery Quiz 5
Due midnight on Thursday, October 8

This week's mastery quiz has nine topics. **Do not answer all nine.** You may answer the two first questions on the newest topics, numbered 9 and 8, and *one* additional topic of the previous three. You may pick one topic you have not yet demonstrated mastery on and answer the question on that topic. (If you are retrying a topic, please complete the entire page.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 10-20 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics:

9. Linear Approximations and Tangent Lines

8. Trigonometry and the Chain Rule

7. Basics of Computing Derivatives

6. Definition of a Derivative

5. Infinite Limits

4. Trigonometric Limits

3. Computing Limits

2. Formal limits

1. Informal limits and continuity

9. **Linear Approximation and Tangent Lines**

(a) Give a formula for a linear approximation of $f(x) = \sqrt{x^3+1}$ near the point $a = 2$.
Solution:

$$f'(x) = \frac{1}{2}(x^3+1)^{-1/2}3x^2$$
$$f'(a) = \frac{1}{2}(9)^{-1/2}\cdot 12 = 2f(x) \qquad = f(a) + f'(a)(x-a) = 3 + 2(x-2).$$

(b) Use your answer in part (a) to estimate $f(2.1)$.
Solution: $f(2.1) \approx 3 + 2(.1) = 3.2.$

(c) Write the equation for the tangent line to $g(x) = \frac{x+2}{x-5}$ at the point $a = 6$.
Solution:

$$g'(x) = \frac{(x-5)-(x+2)}{(x-5)^2}$$
$$g'(6) = \frac{1-8}{1^2} = -7$$
$$y = 8 - 7(x-6)$$

8. **Trigonometry and the Chain Rule** Compute:

(a) $\frac{d}{dx}\cos\left(\frac{\tan(x)+x}{\sqrt{x^2+1}}\right)$
Solution:

$$-\sin\left(\frac{\tan(x)+x}{\sqrt{x^2+1}}\right)\frac{(\sec^2(x)+1)\sqrt{x^2+1} - \frac{1}{2}(x^2+1)^{-1/2}\cdot 2x(\tan(x)+x)}{x^2+1}.$$

(b) $\frac{d}{dx}\sec^3\left(\sqrt[5]{x^3-x}\right)$
Solution: $3\sec^2(\sqrt[5]{x^3-x})\sec(\sqrt[5]{x^3-x})\tan(\sqrt[5]{x^3-x})\frac{1}{5}(x^3-x)^{-4/5}(3x^2-1).$

7. **Basics of Computing Derivatives** Compute the following derivatives while explicitly naming every derivative rule you use.

(a) $\frac{d}{dx}\frac{x^2+1}{\sqrt{x}-x}$
Solution:

$$f'(x) = \frac{(x^2+1)'(\sqrt{x}-x) - (\sqrt{x}-x)'(x^2+1)}{(\sqrt{x}-x)^2} \qquad \text{Quotient Rule}$$
$$= \frac{((x^2)'+1')(\sqrt{x}-x) - (\sqrt{x}'-x')(x^2+1)}{(\sqrt{x}-x)^2} \qquad \text{Sum Rule}$$
$$= \frac{(2x+1')(\sqrt{x}-x) - (\frac{1}{2}x^{-1/2}-1)(x^2+1)}{(\sqrt{x}-x)^2} \qquad \text{Power Rule}$$
$$= \frac{2x(\sqrt{x}-x) - (\frac{1}{2}x^{-1/2}-x)(x^2+1)}{(\sqrt{x}-x)^2} \qquad \text{Constants Rule}$$

(b) $\frac{d}{dx}(5x^4+2x)(x-\sqrt[4]{x}) =$
Solution:

$$\begin{aligned}
\frac{d}{dx}(5x^4+2x)(x-\sqrt[4]{x}) &= (5x^4+2x)'(x-\sqrt[4]{x}) + (x-\sqrt[4]{x})'(5x^4+2x) && \text{Product Rule}\\
&= \left((5x^4)'+(2x)'\right)(x-\sqrt[4]{x}) + (x' - \sqrt[4]{x}')(5x^4+2x) && \text{Sum Rule}\\
&= \left(5(x^4)'+2(x')\right)(x-\sqrt[4]{x}) + (x' - \sqrt[4]{x}')(5x^4+2x) && \text{Scalar Products}\\
&= (20x^3+2)(x-\sqrt[4]{x}) + (1-\frac{1}{4}x^{-3/4})(5x^4+2x) && \text{Power Rule}
\end{aligned}$$

6. **Definition of a Derivative**

Compute the following derivatives, *directly from the formal definition of derivative.*

(a) If $f(x) = \sqrt{x+3}$, find $f'(6)$. **Solution:**

$$\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(6+h)-f(6)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}\\
&= \lim_{h\to 0}\frac{h}{h(\sqrt{9+h}+3)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}\\
&= \lim_{h\to 0}\frac{1}{6}.
\end{aligned}$$

(b) If $g(x) = x^3 - 3x$, find $g'(x)$. **Solution:**

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{(x+h)^3-3(x+h)-x^3+3x}{h}\\
&= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-3x-3h-x^3+3x}{h}\\
&= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h}\\
&= \lim_{h\to 0}3x^2+3xh+h^2-3 = 3x^2-3.
\end{aligned}$$

5. **Infinite Limits** Compute:

(a) $\lim_{x\to -1}\frac{1-x}{1+x} =$
Solution:
The limit of the top is 2 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the denominator can be positive or negative, we cna't be more specific.

(b) $\lim_{x\to+\infty} \frac{x^2-3x+2}{x^3+x-1} =$

Solution:

$$\lim_{x\to+\infty} \frac{x^2-3x+2}{x^3+x-1} = \lim_{x\to+\infty} \frac{1/x-3/x^2+2/x^3}{1+1/x^2-1/x^3} = \frac{0-0+0}{1+0-0} = 0.$$

(c) $\lim_{x\to 4^+} \frac{x+1}{x-4} =$

Solution: The limit of the top is 5 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the bottom will always be positive as we approach from the right, the overall limit is in fact $+\infty$.

4. **Trigonometric Limits**

(a) Show that $\lim_{x\to 2}(x-2)^2\left(1+\sin\left(\frac{2}{(x-2)}\right)\right) = 0$.

Solution: We know that

$$\begin{aligned}
-1 \le \sin\left(\frac{2}{(x-2)}\right) &\le 1 \\
0 \le \sin 1 + \left(\frac{2}{x-2}\right) &\le 2 \\
0 \le (x-2)^2\left(1+\sin\left(\frac{2}{(x-2)}\right)\right) &\le (x-2)^2
\end{aligned}$$

Since $\lim_{x\to 2} 0 = 0$ and $\lim_{x\to 2}(x-2)^2 = 0$, by the Squeeze Theorem, we know that $\lim_{x\to 2}(x-2)^2\left(1+\sin\left(\frac{2}{(x-2)}\right)\right) = 0$.

(b) Compute $\lim_{x\to 1} \frac{\sin(3x-3)\sin(x-1)}{(x-1)^2} =$

Solution:

$$\begin{aligned}
\lim_{x\to 1} \frac{\sin(3x-3)\sin(x-1)}{(x-1)^2} &= \lim_{x\to 1} \frac{\sin(3x-3)}{x-1}\frac{\sin(x-1)}{x-1} \\
&= \lim_{x\to 1} 3\frac{\sin(3x-3)}{3x-3} = 3.
\end{aligned}$$

3. **Computing Limits** Compute:

(a) $\lim_{x\to 3} \frac{1}{x-3} + \frac{3}{x^2-3x} =$

Solution:

$$\begin{aligned}
\lim_{x\to 3} \frac{1}{x-3} + \frac{3}{x^2-3x} &= \lim_{x\to 3} \frac{x^2-3x-3(x-3)}{(x-3)(x^2-3x)} \\
&= \lim_{x\to 3} \frac{x^2-6x+9}{x(x-3)^2} \\
&= \lim_{x\to 3} \frac{1}{x} = \frac{1}{3}.
\end{aligned}$$

(b) $\lim_{x\to -2} \frac{x+2}{\sqrt{x+6}-2} =$
Solution:

$$\lim_{x\to -2} \frac{x+2}{\sqrt{x+6}-2} = \lim_{x\to -2} \frac{(x+2)(\sqrt{x+6}+2)}{x+6-4}$$
$$= \lim_{x\to -2} \frac{\sqrt{x+6}+2}{1} = 4.$$

(c) $\lim_{x\to 1} \frac{x^2-3x+2}{x^2+3x-4} =$
Solution:

$$\lim_{x\to 1} \frac{x^2-3x+2}{x^2+3x-4} = \lim_{x\to 1} \frac{(x-1)(x-2)}{(x+4)(x-1)}$$
$$= \lim_{x\to 1} \frac{x-2}{x+4} = \frac{-1}{5}.$$

2. **Formal Limits**

(a) Write a formal ϵ-δ proof that $\lim_{x\to 1} 2x+4 = 6$.
Solution: Let $\epsilon > 0$ and set $\delta = \epsilon/2$. Then if $0 < |x-1| < \delta$, we have

$$|2x+4-6| = |2x-2| = 2|x-1| < 2\delta = \epsilon.$$

(b) Explicitly naming each limit law you use, compute
$\lim_{x\to 2} \frac{x^2+2x}{4x-3} =$
Solution:

$$\lim_{x\to 2} \frac{x^2+2x}{4x-3} = \frac{\lim_{x\to 2} x^2+2x}{\lim_{x\to 2} 4x-3} \qquad \text{Quotients}$$
$$= \frac{\lim_{x\to 2} x^2 + \lim_{x\to 2} 2x}{\lim_{x\to 2} 4x - \lim_{x\to 2} 3} \qquad \text{Sums}$$
$$= \frac{\lim_{x\to 2} x^2 + \lim_{x\to 2} 2 \lim_{x\to 2} x}{\lim_{x\to 2} 4 \lim_{x\to 2} x - \lim_{x\to 2} 3} \qquad \text{products}$$

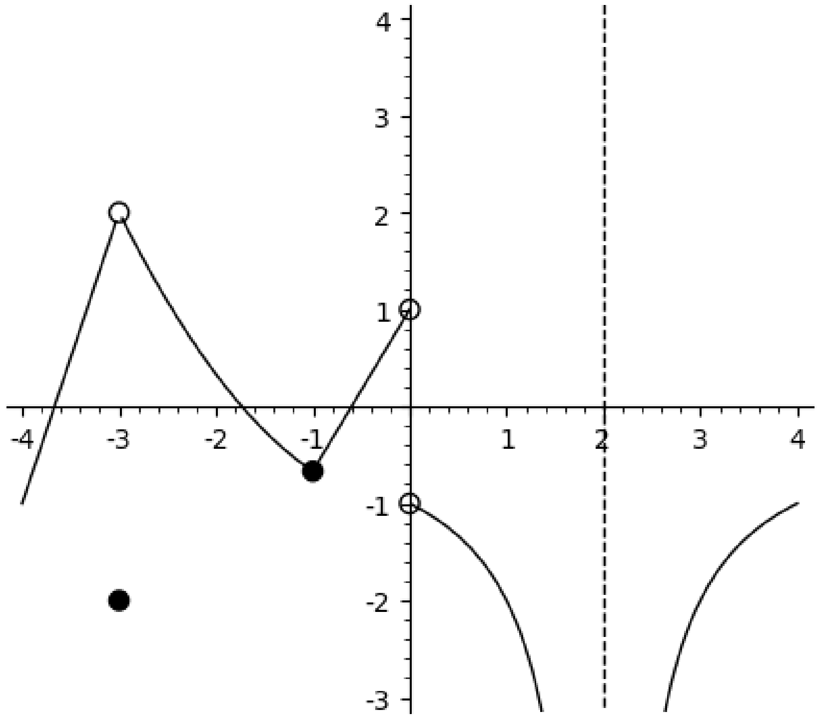
[could also use scalars]

$$= \frac{\lim_{x\to 2} x^2 + 2\lim_{x\to 2} x}{4\lim_{x\to 2} x - 3} \qquad \text{constants}$$
$$= \frac{(\lim_{x\to 2} x)^2 + 2\lim_{x\to 2} x}{4\lim_{x\to 2} x - 3} \qquad \text{Exponents}$$
$$= \frac{2^2 + 2\cdot 2}{4\cdot 2 - 3} = \frac{8}{5} \qquad \text{identity.}$$

1. **Informal Continuity and Limits**

(a) Give a (zeroth-order) approximate value for $\sqrt{16.3}$, and explain how you got it.
Solution: We know that 16.3 is close to 16, so we expect $\sqrt{16.3}$ to be close to $\sqrt{16} = 4$.

Here is the graph of a function f:



For each of the following questions, if your answer is "does not exist", explain in a few words why it does not exist. If your answer is just a number, you don't need to explain.

(b) What is the domain of f?

Solution: All reals except $0, 2$. I'd also except all of $[-4, 4]$ except $0, 2$.

(c) Where (if anywhere) is f discontinuous?

Solution: $x = -3, 0, 2$.

(d) What is $\lim_{x \to -3} f(x)$?

Solution: 2

(e) What is $f(-3)$?

Solution: -2

(f) What is $\lim_{x \to 0^+} f(x)$?

Solution: This limit is -1.

(g) What is $\lim_{x \to 2} f(x)$?

Solution: $-\infty$