Math 1231 Fall 2020
Single-Variable Calculus I Mastery Quiz 6
Due midnight on Thursday, October 15

This week's mastery quiz has nine topics. **Do not answer all nine.** You may answer the two first questions on the newest topics, numbered 11 and 10, and *one* additional topic of the previous three. You may pick one topic you have not yet demonstrated mastery on and answer the question on that topic. (If you are retrying a topic, please complete the entire page.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 10-20 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics:

11. Implicit Differentiation
10. Rates of Change
9. Linear Approximations and Tangent Lines
8. Trigonometry and the Chain Rule
7. Basics of Computing Derivatives
6. Definition of a Derivative
5. Infinite Limits
3. Computing Limits
2. Formal limits

11. **Implicit Differentiation**

(a) Find an equation for the tangent line to the curve $x^3y + 3xy^2 = 40$ at the point $(2, 2)$.

Solution:

$$\begin{aligned} 3x^2y + x^3y' + 3y^2 + 6xyy' &= 0 \\ 24 + 8y' + 12 + 24y' &= 0 \\ 32y' &= -36 \\ y' &= -9/8 \end{aligned}$$

so the equation for the tangent line is

$$y - 2 = \frac{-9}{8}(x - 2).$$

(b) Find a formula for y' in terms of x and y if $\sqrt{x+y} = x^3y^2$.

Solution:

$$\begin{aligned} \frac{1}{2}(x+y)^{-1/2}(1+y') &= 3x^2y^2 + 2x^3yy' \\ \frac{1}{2}(x+y)^{-1/2}y' - 2x^3yy' &= 3x^2y^2 - \frac{1}{2}(x+y)^{-1/2} \\ y' &= \frac{3x^2y^2 - \frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1.2} - 2x^3y}. \end{aligned}$$

10. **Rates of Change**

(a) Let $F(x) = 1/x + 1$ be the amount of pressure exerted on a beam in pounds per square inch at a point x inches to the right of its left end.

(i) What does the derivative $F'(x)$ represent, and what are its units?
Solution: The derivative $F'(x)$ is the rate at which pressure is increasing as you move to the right along the stick. Its units are pounds per square inch per inch, or pounds per cubic inch.

(ii) Compute $F'(5)$. What does this tell you?
Solution: $F'(x) = -1/x^2$ so $F'(5) = -1/25$. This means that if we are five inches to the right of the endpoint, moving one more inch to the right should decrease the pressure by about $1/25$ of a pound per square inch.

(b) Suppose the height of a particle in centimeters is given as a function of time in seconds $p(t) = t^3 - 3t$.

(i) When is the velocity zero?
Solution: $p'(t) = 3t^2 - 3$ is zero when $t = \pm 1$ second.

(ii) When is the acceleration zero?
Solution: $p''(t) = 6t$ is zero when $t = 0$ seconds.

9. **Linear Approximation and Tangent Lines**

(a) Give a formula for a linear approximation of $f(x) = \frac{x}{x-3}$ near the point $a = 4$.
Solution:

$$f'(x) = \frac{(x-3)-x}{(x-3)^2} f'(4) \quad = \frac{-3}{1^2} = -3 f(x) \approx f(a) + f'(a)(x-a) = 4 - 3(x-4).$$

(b) Use your answer in part (a) to estimate $f(3.9)$.
Solution: $f(3.9) \approx 4 - 3(-.1) = 4.3$.

(c) Write the equation for the tangent line to $g(x) = \sin(x^2 - 3x)$ at the point $a = 0$.
Solution:

$$g'(x) = \cos(x^2 - x)(2x-3) g'(0) \qquad = 1 \cdot (0-3) = -3 y = 0 - 3(x-0)$$

8. **Trigonometry and the Chain Rule** Compute:

(a) $\frac{d}{dx} \sec(\tan(\cos((x+1)^2)))$
Solution:

$$\sec(\tan(\cos((x+1)^2)))\tan(\tan(\cos((x+1)^2)))\sec^2(\cos((x+1)^2))(-\sin((x+1)^2))2(x+1)$$

(b) $\frac{d}{dx} \frac{\sqrt{x^3+x} + \sqrt[3]{x}}{\csc(x)+1}$
Solution:

$$\frac{\left(\frac{1}{2}(x^3+x)^{-1/2}(3x^2+1) + \frac{1}{3}x^{-2/3}\right)(\csc(x)+1) - (-\csc(x)\cot(x))(\sqrt{x^3+x} + \sqrt[3]{x})}{(\csc(x)+1)^2}$$

7. **Basics of Computing Derivatives** Compute the following derivatives while explicitly naming every derivative rule you use.

(a) $\frac{d}{dx}(x^2+1)(x-3)$
Solution:

$$\begin{aligned} f'(x) &= (x^2+1)'(x-3) + (x^2+1)(x-3)' && \text{Product Rule} \\ &= (\frac{d}{dx}x^2 + \frac{d}{dx}1)(x-3) + (x^2+1)(\frac{d}{dx}x - \frac{d}{dx}3) && \text{Sum Rule} \\ &= (\frac{d}{dx}x^2 + 0)(x-3) + (x^2+1)(\frac{d}{dx}x - 0) && \text{Constants} \\ &= (2x)(x-3) + (x^2+1)(1) && \text{Power Rule} \end{aligned}$$

(b) $\frac{d}{dx} \frac{x^2+3x}{x^3-1} =$

Solution:

$$\begin{aligned}
\frac{d}{dx}\frac{x^2+3x}{x^3-1} &= \frac{(x^2+3x)'(x^3-1)-(x^3-1)'(x^2+3x)}{(x^3-1)^2} && \text{Quotient Rule}\\
&= \frac{((x^2)'+(3x)')(x^3-1)-((x^3)'-1')(x^2+3x)}{(x^3-1)^2} && \text{Sum Rule}\\
&= \frac{((x^2)'+3(x)')(x^3-1)-((x^3)'-1')(x^2+3x)}{(x^3-1)^2} && \text{Scalar Products}\\
&= \frac{((x^2)'+3(x)')(x^3-1)-((x^3)')(x^2+3x)}{(x^3-1)^2} && \text{Constants Rule}\\
&= \frac{(2x+3)(x^3-1)-(3x^2)(x^2+3x)}{(x^3-1)^2} && \text{Power Rule}
\end{aligned}$$

6. **Definition of a Derivative**

   Compute the following derivatives, *directly from the formal definition of derivative.*

   (a) If $f(x) = \frac{x+1}{x-1}$, find $f'(2)$. **Solution:**

$$\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\\
&= \lim_{h\to 0}\frac{\frac{3+h}{1+h}-3}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\frac{3+h-3(1+h)}{1+h}\\
&= \lim_{h\to 0}\frac{-2h}{h(1+h)}\\
&= \lim_{h\to 0}\frac{-2}{1+h} = -2.
\end{aligned}$$

   (b) If $g(x) = \sqrt{x-5}$, find $g'(x)$. **Solution:**

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{x+h-5}-\sqrt{x-5}}{h}\\
&= \lim_{h\to 0}\frac{x+h-5-(x-5)}{h(\sqrt{x+h-5}+\sqrt{x-5})}\\
&= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h-5}+\sqrt{x-5})}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h-5}+\sqrt{x-5}} = \frac{1}{2\sqrt{x-5}}.
\end{aligned}$$

5. **Infinite Limits** Compute:

   (a) $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+x+1}}{x+3} =$

Solution:

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+x+1}}{x+3}=\lim_{x\to-\infty}\frac{-\sqrt{1+1/x+1/x^2}}{1+3/x}=\frac{-\sqrt{1+0+0}}{1+0}=-1.$$

(b) $\lim_{x\to3}\frac{x+3}{x^2-9}=$
Solution:
The limit of the top is 6 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the denominator can be positive or negative, we can't be more specific.

(c) $\lim_{x\to2}\frac{x-1}{(x-2)^2}=$
Solution: The limit of the top is 1 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the bottom will always be positive since it's a square, the overall limit is in fact $+\infty$.

3. **Computing Limits** Compute:

(a) $\lim_{x\to1}\frac{\sqrt{x+3}-2}{x-1}=$
Solution:

$$\begin{aligned}\lim_{x\to1}\frac{\sqrt{x+3}-2}{x-1}&=\lim_{x\to1}\frac{x+3-2}{(x-1)(\sqrt{x+3}+2)}\\&=\lim_{x\to1}\frac{x-1}{(x-1)(\sqrt{x+3}+2)}\\&=\lim_{x\to1}\frac{1}{\sqrt{x+3}+2}=\frac{1}{4}.\end{aligned}$$

(b) $\lim_{x\to3}\frac{x^2-5x+6}{x^2-x-6}=$
Solution:

$$\begin{aligned}\lim_{x\to3}\frac{x^2-5x+6}{x^2-x-6}&=\lim_{x\to3}\frac{(x-3)(x-2)}{(x-3)(x+2)}\\&=\lim_{x\to3}\frac{x-2}{x+2}=1/5.\end{aligned}$$

(c) $\lim_{x\to0}\frac{1}{x}-\frac{1}{x^2+x}=$
Solution:

$$\begin{aligned}\lim_{x\to0}\frac{x+1-1}{x^2+x}&=\lim_{x\to0}\frac{x}{x^2+x}\\&=\lim_{x\to1}\frac{1}{x+1}=1/2.\end{aligned}$$

2. **Formal Limits**

(a) Write a formal ϵ-δ proof that $\lim_{x\to 3} 3x - 1 = 8$.

Solution: Let $\epsilon > 0$ and set $\delta = \epsilon/3$. Then if $0 < |x-3| < \delta$, we have

$$|3x - 1 - 8| = |3x - 9| = 3|x - 3| < 3\delta = \epsilon.$$

(b) Explicitly naming each limit law you use, compute

$\lim_{x\to -1} \frac{\sqrt{x^2+3}}{2x+3} =$

Solution:

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x^2+3}}{2x+3} &= \frac{\lim_{x\to -1}\sqrt{x^2+3}}{\lim_{x\to -1} 2x+3} && \text{Quotients}\\
&= \frac{\sqrt{\lim_{x\to -1} x^2+3}}{\lim_{x\to -1} 2x+3} && \text{Exponents}\\
&= \frac{\sqrt{\lim_{x\to -1} x^2+\lim_{x\to -1} 3}}{\lim_{x\to -1} 2x+\lim_{x\to -1} 3} && \text{Sums}\\
&= \frac{\sqrt{\lim_{x\to -1} x^2+\lim_{x\to -1} 3}}{2\lim_{x\to -1} x+\lim_{x\to -1} 3} && \text{scalar products}\\
& && \text{[could also use regular product]}\\
&= \frac{\sqrt{(\lim_{x\to -1} x)^2+\lim_{x\to -1} 3}}{2\lim_{x\to -1} x+\lim_{x\to -1} 3} && \text{Exponents}\\
&= \frac{\sqrt{(\lim_{x\to -1} x)^2+3}}{2\lim_{x\to -1} x+3} && \text{Constants}\\
&= \frac{\sqrt{(-1)^2+3}}{2(-1)+3} = \frac{\sqrt{4}}{1} = 2 && \text{Identity}
\end{aligned}$$