Math 1231 Fall 2020
Single-Variable Calculus I Mastery Quiz 8
Due midnight on Thursday, October 29

This week's mastery quiz has ten topics. **Do not answer all ten.** You may answer the two first questions on the newest topics, numbered 13 and 12, and *one* additional topic. You may pick one topic you have not yet demonstrated mastery on and answer the question on that topic. (If you are retrying a topic, please complete the entire page.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 10-20 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics:

13. Global Maxima and Critical Points
12. Related Rates
11. Implicit Differentiation
10. Rates of Change
9. Linear Approximations and Tangent Lines
8. Trigonometry and the Chain Rule
6. Definition of a Derivative
4. Trigonometric Limits
3. Computing Limits
1. Informal limits and continuity

13. **Global Maxima and Critical Points**

(a) Find the absolute extrema of $g(x) = 3x^4 - 2x^3 - 3x^2 + 5$ on the interval $[-1, 2]$, and justify your claim that these are the absolute extrema.

Solution:

g is continuous on the closed interval $[-1, 2]$, so by the Extreme Value Theorem it has a maximum and a minimum on the interval. This must happen at a critical point or an endpoint. (This argument is necessary! Otherwise there's no reason to expect the largest local max to be a global max.)

We compute

$$g'(x) = 12x^3 - 6x^2 - 6x = 6x(2x^2 - x - 1) = 6x(2x + 1)(x - 1)$$

which is zero at $0, 1, -1/2$. Then we compute

$$\begin{aligned} g(-1) &= 7 \\ g(-1/2) &= \frac{3}{16} + \frac{1}{4} - \frac{3}{4} + 5 = 5 - \frac{5}{16} = \frac{75}{16} = 4.6875 \\ g(0) &= 5 \\ g(1) &= 3 \\ g(2) &= 48 - 16 - 12 + 5 = 25. \end{aligned}$$

Thus g has an absolute maximum of 25 at 2, and an absolute minimum of 3 at 1.

(b) Find all the critical points of $f(x) = \sqrt[3]{x^3 - 3x}$.

Solution:

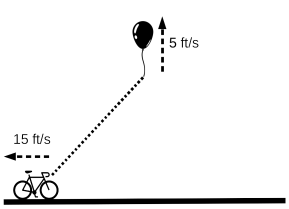
We have

$$f'(x) = \frac{1}{3}(x^3 - 3x)^{-2/3}(3x^2 - 3) = \frac{(x-1)(x+1)}{\sqrt[3]{x(x^2-3)}^2}.$$

This is zero when $x = \pm 1$ and is undefined when $x = 0$ or $x = \pm\sqrt{3}$. Thus the critical points are $-\sqrt{3}, -1, 0, 1, \sqrt{3}$.

12. **Related Rates**

A balloon is rising at a constant speed of 5 feet per second. A boy is cycling along a straight road at a speed of 15 feet per second. When he passes under the balloon, it is 45 feet above him. How fast is the distance between the boy and the balloon increasing 3 seconds later?



Solution:

We see that the height of the balloon is $h = 60$ feet, and the derivative is $h' = 5$ feet per second. The distance between the boy and the point under the balloon is $w = 45$

feet and the derivative is $w' = 15$ feet per second. The distance between them is given by $d^2 = w^2 + h^2$, and so we can compute first that the current distance is 75 feet, and then that

$$\begin{aligned} 2dd' &= 2ww' + 2hh' \\ dd' &= ww' + hh' \\ 75d' &= 45 \cdot 15 + 60 \cdot 5 = 675 + 300 = 975 \\ d' &= \frac{975}{75} = \frac{325}{25} = 13. \end{aligned}$$

Thus the distance is increasing by 13 feet per second.

11. **Implicit Differentiation**

(a) Find an equation for the tangent line to the curve $\sin(x^2y) + \cos(xy) = 1$ at the point $(1, \pi/2)$.
Solution:

$$\begin{aligned} \cos(x^2y)(2xy + x^2y') - \sin(xy)(y + xy') &= 0 \\ 0 - (\pi/2 + y') &= 0 \\ y' &= -\pi/2 \end{aligned}$$

so the equation for the tangent line is

$$y - \pi/2 = -\pi/2(x - 1).$$

(b) Find a formula for y' in terms of x and y if $x^3y + x^2y^2 + y^4 = 0$.
Solution:

$$\begin{aligned} 3x^2y + x^3y' + 2xy^2 + 2x^2yy' + 4y^3y' &= 0 \\ x^3y' + 2x^2yy' + 4y^3y' &= -3x^2y - 2xy^2 \\ y' &= -\frac{3x^2y + 2xy^2}{x^3 + 2x^2y + 4y^3}. \end{aligned}$$

10. **Rates of Change**

(a) The *area moment of inertia* of a steel beam measures how difficult it is to bend, and is measured in m^4. If a square beam has a side length of s meters, then its moment of inertia is given by $L(s) = s^4/12$.

(i) What does the derivative $L'(s)$ represent physically, and what are its units?
Solution:
$L'(s)$ describes how much increasing the side length by a meter would increase the area moment of inertia. Its units are $\frac{m^4}{m} = m^3$.

(ii) Compute $L'(3)$. What does this tell you physically? **Solution:**
$L'(s) = 4s^3/12 = s^3/3$ so $L'(3) = 27/3 = 9m^3$. This tells us that if we increase the side length by one meter from 3 meters to 4 meters, we should increase the moment of inertia by about 9 m^4.

(b) Suppose the vertical position of a weight on a spring in inches is given as a function of time in seconds by the formula $h(t) = \cos(3t)$.

   (i) When is the velocity zero?
   Solution: $p'(t) = -3\sin(3t)$ so the velocity is zero when $\sin(3t) = 0$. This happens when $3t = 0, \pi, 2\pi, \ldots$, and thus when $t = 0, \pi/3, 2\pi/3, \pi, \ldots$. In other words, at $n\pi/3$.

   (ii) When is the acceleration zero?
   Solution: $p''(t) = -9\cos(3t)$ is zero when $3t = \pi/2, 3\pi/2, \ldots$, and thus when $t = \pi/6, 3\pi/6, 5\pi/6, \ldots$. We could say $t = (2n+1)\pi/6$.

9. **Linear Approximation and Tangent Lines**

(a) Give a formula for a linear approximation of $f(x) = x\sqrt{x+1}$ near the point $a = 3$.
Solution:

$$\begin{aligned} f'(x) &= \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} \\ f'(3) &= 2 + \frac{3}{4} = \frac{11}{4} \\ f(x) &\approx f(a) + f'(a)(x-a) = 6 + \frac{11}{4}(x-3). \end{aligned}$$

(b) Use your answer in part (a) to estimate $f(3.2)$.
Solution: $f(3.2) \approx 6 + \frac{11}{4}(.2) = 6 + \frac{11}{20} = \frac{132}{20}$.

(c) Write the equation for the tangent line to $g(x) = 2x - \tan(x)$ at the point $a = \pi$.
Solution:

$$\begin{aligned} g(\pi) &= 2\pi - 0 = \pi \\ g'(x) &= 2 - \sec^2(x) \\ g'(\pi) &= 2 - 1 = 1 \\ y &= 2\pi + (x - \pi) \end{aligned}$$

8. **Trigonometry and the Chain Rule** Compute:

(a) $\dfrac{d}{dx}\left(\tan\left(\sec\left(\dfrac{x+1}{x^3-4}\right)\right)\right)^{3/5}$
Solution:

$$\frac{3}{5}\tan^{-2/5}\left(\sec\left(\frac{x+1}{x^3-4}\right)\right)\sec^2\left(\sec\left(\frac{x+1}{x^3-4}\right)\right) \cdot \sec\left(\frac{x+1}{x^3-4}\right)\tan\left(\frac{x+1}{x^3-4}\right)\frac{(x^3-4) - 3x^2(x+1)}{(x^3-4)^2}.$$

(b) $\dfrac{d}{dx}\dfrac{\sin(\csc(x^2+1))}{x^4+\cos(x)}$
Solution:

$$\frac{\big(\cos(\csc(x^2+1))(-\csc(x^2+1)\cot(x^2+1))2x\big)(x^4+\cos(x)) - (4x^3 - \sin(x))\sin(\csc(x^2+1))}{(x^4+\cos(x))^2}.$$

6. **Definition of a Derivative**

Compute the following derivatives, *directly from the formal definition of derivative.*

(a) If $f(x) = 3x^2 - x$, find $f'(-3)$. **Solution:**

$$\begin{aligned} f'(-3) &= \lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h} \\ &= \lim_{h\to 0} \frac{3(h-3)^2 - (h-3) - 30}{h} \\ &= \lim_{h\to 0} \frac{3h^2 - 18h + 27 - h + 3 - 30}{h} \\ &= \lim_{h\to 0} \frac{3h^2 - 19h}{h} \\ &= \lim_{h\to 0} 3h - 19 = -19. \end{aligned}$$

(b) If $g(x) = \frac{x}{x+2}$, find $g'(x)$. **Solution:**

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{x+h}{x+h+2} - \frac{x}{x+2}}{h} \\ &= \lim_{h\to 0} \frac{(x+h)(x+2) - x(x+h+2)}{h(x+2)(x+h+2)} \\ &= \lim_{h\to 0} \frac{x^2 + hx + 2x + 2h - x^2 - xh - 2x}{h(x+2)(x+h+2)} \\ &= \lim_{h\to 0} \frac{2h}{h(x+2)(x+h+2)} \\ &= \lim_{h\to 0} \frac{2}{(x+2)(x+h+2)} = \frac{2}{(x+2)^2}. \end{aligned}$$

4. **Trigonometric Limits**

(a) Show that $\lim_{x\to 0} x \sin\left(\frac{3}{x}\right) = 0.$

Solution: We know that

$$-1 \le \sin\left(\frac{3}{x}\right) \le 1$$
$$-|x| \le x\sin\left(\frac{3}{x}\right) \le |x|$$

Since $\lim_{x\to 0} -|x| = 0$ and $\lim_{x\to 0} |x| = 0$, by the Squeeze Theorem, we know that $\lim_{x\to 0} x\sin\left(\frac{3}{x}\right) = 0$.

(b) Compute $\lim_{x\to 0} \frac{x\sin(2x)}{\sin(5x)\sin(3x)} =$

Solution:

$$\lim_{x\to 0} \frac{x\sin(2x)}{\sin(5x)\sin(3x)} = \lim_{x\to 0} \frac{5x}{\sin(5x)} \frac{\sin(2x)}{2x} \frac{3x}{\sin(3x)} \frac{2}{15} = \frac{2}{15}.$$

3. **Computing Limits** Compute:

(a) $\lim_{x\to 2} \frac{3}{x-2} - \frac{6}{x(x-2)} =$

Solution:

$$\begin{aligned}\lim_{x\to 2} \frac{3}{x-2} - \frac{6}{x(x-2)} &= \lim_{x\to 2} \frac{3x-6}{x(x-2)} \\ &= \lim_{x\to 2} \frac{3}{x} = 3/2.\end{aligned}$$

(b) $\lim_{x\to 3} \frac{x^2-5x+4}{x^2+3x+2} =$

Solution:

$$\lim_{x\to 3} \frac{x^2-5x+4}{x^2+3x+2} = \frac{9-15+4}{9+9+2} = \frac{-2}{20}.$$

(c) $\lim_{x\to -3} \frac{\sqrt{x+12}-3}{x+3} =$

Solution:

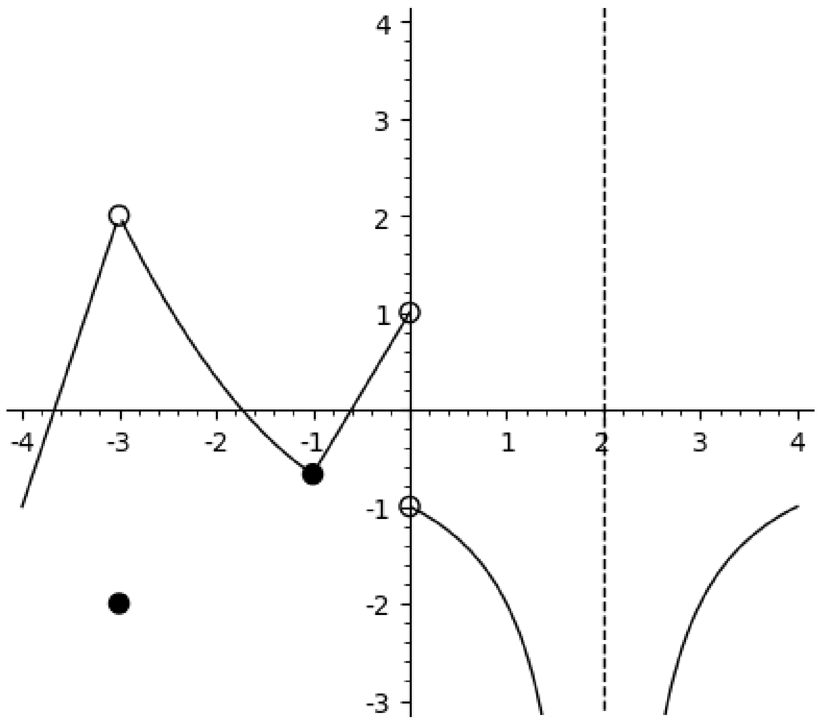
$$\begin{aligned}\lim_{x\to -3} \frac{\sqrt{x+12}-3}{x+3} &= \lim_{x\to -3} \frac{x+12-9}{(x+3)(\sqrt{x+12}+3)} \\ &= \lim_{x\to -3} \frac{1}{\sqrt{x+12}+3} = \frac{1}{6}.\end{aligned}$$

1. **Informal Continuity and Limits**

(a) Give a (zeroth-order) approximate value for $\sqrt{16.3}$, and explain how you got it.

Solution: We know that 16.3 is close to 16, so we expect $\sqrt{16.3}$ to be close to $\sqrt{16} = 4$.

Here is the graph of a function f:



For each of the following questions, if your answer is "does not exist", explain in a few words why it does not exist. If your answer is just a number, you don't need to explain.

(b) What is the domain of f?
Solution: All reals except $0, 2$. I'd also except all of $[-4, 4]$ except $0, 2$.

(c) Where (if anywhere) is f discontinuous?
Solution: $x = -3, 0, 2$.

(d) What is $\lim_{x \to -3} f(x)$?
Solution: 2

(e) What is $f(-3)$?
Solution: -2

(f) What is $\lim_{x \to 0^+} f(x)$?
Solution: This limit is -1.

(g) What is $\lim_{x \to 2} f(x)$?
Solution: $-\infty$