Math 1231 Fall 2020
Single-Variable Calculus I Mastery Quiz 9
Due midnight on Thursday, November 5

This week's mastery quiz has eight topics. **Do not answer all ten.** You may answer the two first questions on the newest topics, numbered 15 and 14, and *one* additional topic. You may pick one topic you have not yet demonstrated mastery on and answer the question on that topic. (If you are retrying a topic, please complete the entire page.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 10-20 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics:

15. Curve Sketching
14. First and Second Derivative Tests
13. Global Maxima and Critical Points
12. Related Rates
11. Implicit Differentiation
10. Rates of Change
4. Trigonometric Limits
1. Informal limits and continuity

15. **Curve Sketching**

Let $f(x) = \dfrac{(x-2)^2}{x-1}$. We can compute that

$$f'(x) = \frac{x(x-2)}{(x-1)^2}$$
$$f''(x) = \frac{2}{(x-1)^3}.$$

Sketch a graph of f. Your answer should discuss the domain, roots, asymptotes, limits at infinity, critical points and values, intervals of increase and decrease, and concavity and points of inflection.

Solution:

The function is defined for all real numbers except 1. We compute that $\lim_{x\to 1^+} f(x) = +\infty$ and $\lim_{x\to 1^-} f(x) = -\infty$. There is a root of f at $x = 2$, and we compute that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

$f'(x)$ is undefined at $x = 1$ and is 0 at $0, 2$ so the critical points are $0, 1, 2$. We compute that $f(0) = -4$ and $f(2) = 0$. We make a chart:

	x	$x-2$	$(x-1)^{-2}$	$f'(x)$
$x < 0$	$-$	$-$	$+$	$+$
$0 < x < 1$	$+$	$-$	$+$	$-$
$1 < x < 2$	$+$	$-$	$+$	$-$
$2 < x$	$+$	$+$	$+$	$+$

Thus f is increasing on $(-\infty, 0)$ and $(2, +\infty)$, and is decreasing on $(0, 2)$. It has relative a relative maximum at $(0, -4)$ and a relative minimum at $(2, 0)$; it doens't have a value at 1.

$f''(x)$ is undefined at 1 and is never 0, so the only possible point of inflection is 1. We see that $f''(x)$ is negative if $x < 1$ and positive if $x > 1$, so f is concave down on $(-\infty, 1)$ and concave up on $(1, +\infty)$.

14. **First and Second Derivative Tests**

(a) Classify the critical points and relative extrema of $f(x) = 5 + 8x^3 + x^4$.

Solution:

We have $f'(x) = 24x^2 + 4x^3 = 4x^2(6 + x)$. So the critical points are $-6, 0$. We could try the second derivative test: we get $f''(x) = 48x + 12x^2$. Then $f''(-6) = 12(-24 + 36) = 144 > 0$, so this is a relative minimum. But $f''(0) = 0$ which doesn't tell us anything. We have to pass to the first derivative test.

We make a chart:

	$4x^2$	$6+x$	$f'(x)$
$x < -6$	$+$	$-$	$-$
$-6 < x < 0$	$+$	$+$	$+$
$0 < x$	$+$	$+$	$+$

Thus f is decreasing when $x < -6$ and increasing when $x > -6$, and so we see that f has a minimum at $x = -6$, and a point which is neither a maximum nor a minimum at $x = 0$,.

(b) Classify the critical points and relative extrema of $g(x) = \frac{2x-1}{x^2+2}$.

Solution: We have

$$g'(x) = \frac{2(x^2+2) - 2x(2x-1)}{(x^2+2)^2} = \frac{-2x^2+2x+4}{(x^2+2)^2}$$
$$= -2\frac{x^2-x-2}{(x^2+2)^2} = -2\frac{(x-2)(x+1)}{(x^2+2)^2}$$

so the critical points are 2 and -1. (The derivative is defined everywhere).

To classify these critical points we need to use either the first or second derivative test. I think the first derivative test looks easier here, purely because I don't want to compute the second derivative. I get the table

	$x-2$	$x+1$	$\frac{-2}{(x^2+2)^2}$	$g'(x)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 2$	$-$	$+$	$-$	$+$
$2 < x$	$+$	$+$	$-$	$-$

Thus we see that there is a relative minimum at -1 and a relative maximum at 2.

But we could use the second derivative test if we really wanted to. We compute

$$g''(x) = -2\frac{(2x-1)(x^2+2)^2 - 2(x^2+2)2x(x^2-x-2)}{(x^2+2)^4}$$
$$g''(-1) = -2\frac{(-3)(3)^2 - 2(3)(-2)(0)}{3^4} = \frac{-2\cdot(-27)}{3^4} = 2/3 > 0$$
$$g''(2) = -2\frac{3(6)^2 - 2(6)4(0)}{6^4} = \frac{-1}{6} < 0.$$

Thus $g''(-1) > 0$ so g has a minimum at -1; and $g''(2) < 0$ so g has a maximum at 2.

13. **Global Maxima and Critical Points**

(a) Find the absolute extrema of $g(x) = x^3 - 3x^2 - 9x + 5$ on $[-2, 4]$, and justify your claim that these are in fact absolute extrema.

Solution: g is continuous on the closed interval $[-2, 4]$ so by the extreme value theorem it has an absolute maximum and an absolute minimum.

We compute $g'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$ is always defined, and is zero if $x = -1$ or $x = 3$. So the critical points are -1 and 3, and we need to check the points $-2, -1, 3, 4$.

$$g(-2) = 3 \qquad g(-1) = 10$$
$$g(3) = -22 \qquad g(4) = -15.$$

Thus g has a maximum of 10 at -1 and a minimum of -22 at 3.

(b) Find all the critical points of $g(x) = \frac{x^2-3x-4}{x+5}$ **Solution:**

$$\begin{aligned} g'(x) &= \frac{(2x-3)(x+5)-(x^2-3x-4)}{(x+5)^2} \\ &= \frac{x^2+10x-11}{(x+5)^2} \\ &= \frac{(x+11)(x-1)}{(x+5)^2}. \end{aligned}$$

Thus $g'(x) = 0$ when $x = -11$ or $x = 1$, and $g'(x)$ is undefined when $x = -5$. So the critical points are $1, -5, -11$.

12. **Related Rates**

A rocket is taking off with a perfectly vertical path, and is being tracked by a radar station on the ground four miles from the launch pad. How fast is the rocket rising when it is three miles high and its distance from the radar station is increasing at a rate of 3000 miles per hour.

Solution: We know one speed and want to know another, and we also know distances. This means we probably want to use the distance formula and take its derivative to find speeds.

We write $h = 3$ miles, and can work out that $d = 5$ miles. We know that $d' =$ miles per hours.

We know that $d^2 = h^2 + 4^2$ and thus $2dd' = 2hh'$. Plugging in values gives us

$$\begin{aligned} 2 \cdot 5mi \cdot 3000mi/hr &= 2 \cdot 3mi \cdot h' \\ h' &= 5000mi/hr. \end{aligned}$$

Thus the rocket is rising at 5000 miles per hour.

11. **Implicit Differentiation**

(a) Write a tangent line to the curve $x^2y = x + 2y$ at the point $(2, 1)$.
Solution:

$$\begin{aligned} 2xy + x^2y' &= 1 + 2y' \\ (x^2-2)y' &= 1 - 2xy \\ y' &= \frac{1-2xy}{x^2-2} \\ y'(2,1) &= \frac{1-4}{4-2} = \frac{-3}{2} \end{aligned}$$

or

$$\begin{aligned} 2xy + x^2y' &= 1 + 2y' \\ 4 + 4y' &= 1 + 2y' \\ 2y' &= -3 \\ y' &= -3/2 \end{aligned}$$

so the equation for the line is

$$y - 1 = \frac{-3}{2}(x - 2).$$

(b) Find a formula for y' in terms of x and y if $xy = x^2 \sin(y)$.
Solution:

$$\begin{aligned} y + xy' &= 2x\sin(y) + x^2\cos(y)y' \\ y - 2x\sin(y) &= x^2\cos(y)y' - xy' \\ \frac{y - 2x\sin(y)}{x^2\cos(y) - x} &= y' \end{aligned}$$

10. **Rates of Change**

(a) The force a magnet exerts on a piece of iron depends on the distance between the magnet and the metal. Let $F(d) = \frac{2}{d^2}$ give the force exerted by the magnet in Newtons, where d is the distance between them in meters.

(i) What does the derivative $F'(d)$ represent, and what are its units?
Solution: The derivative is the rate at which the amount of force changes as you change the distance between the magnet and the iron; its units are Netwons per meter.

(ii) Calculate $F'(2)$. What does this tell you physically?
Solution: $F'(d) = \frac{-4}{d^3}$ so $F'(3) = \frac{-4}{8} = -1/2$. This means that moving the iron another meter away from the magnet should reduce the force by about half a Newton.

(b) Suppose the distance between two particles in centimeters is given as a function of time in seconds by the formula $d(t) = t + \frac{1}{t}$.

(i) When is the velocity zero?
Solution: $d'(t) = 1 - 1/t^2$ so the velocity is zero when $t = \pm 1$.

(ii) When is the acceleration zero?
Solution: $d''(t) = 2/t^3$ is never zero.

9. **Linear Approximation and Tangent Lines**

(a) Use a linear approximation to estimate $\sqrt[3]{8.3}$.
Solution: Use $f(x) = \sqrt[3]{x}$ and $a = 8$. Then we have $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(a) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$, and

$$\begin{aligned} f(x) f(8) + &\approx f'(8)(x - 8) \\ f(8.3) &\approx 2 + \frac{1}{12}(8.3 - 8) = 2 + \frac{1}{40} = \frac{81}{40}. \end{aligned}$$

(b) Find an equation of the line tangent to $y = x\tan x$ at the point $x = \pi/4$.
Solution: $y' = \tan x + x\sec^2 x$ so $y'(\pi/4) = 1 + \pi/2$, and thus the equation for the tangent line is $y - \pi/4 = (1 + \pi/2)(x - \pi/4)$.

4. **Trigonometric Limits**

(a) Show that $\lim_{x \to 3} |x-3| \cos\left(\frac{x}{x-3}\right) = 0$.

Solution: We know that

$$-1 \leq \cos\left(\frac{x}{x-3}\right) \leq 1$$
$$-|x-3| \leq |x-3| \sin\left(\frac{x}{x-3}\right) \leq |x-3|$$

Since $\lim_{x\to 3} -|x-3| = 0$ and $\lim_{x\to 3} |x-3| = 0$, by the Squeeze Theorem, we know that $\lim_{x\to 3} |x-3| \sin\left(\frac{x}{x-3}\right) = 0$.

(b) Compute $\lim_{x \to 0} \frac{\sin(x)\sin(2x)\sin(3x)}{x^3}$.

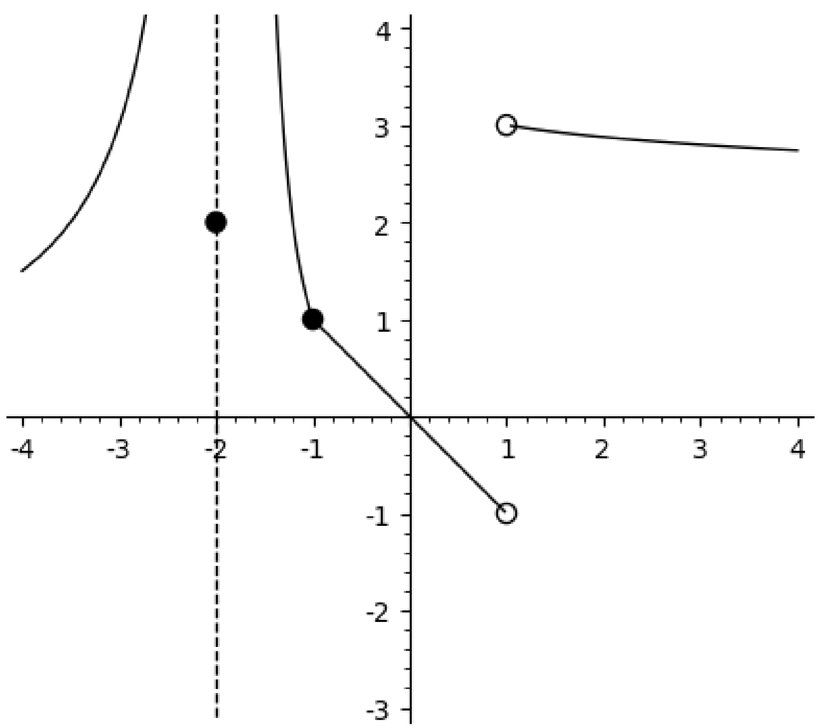
Solution:

1. **Informal Continuity and Limits**

(a) Give a (zeroth-order) approximate value for cos(3), and explain how you got it.

Solution: We know that 3 is close to π, so $\cos(3) \approx \cos(\pi) = -1$.

Here is the graph of a function f:



For each of the following questions, if your answer is "does not exist", explain in a few words why it does not exist. If your answer is just a number, you don't need to explain.

(b) What is the domain of f?

Solution: All reals except 1. I'd also except all of $[-4, 4]$ except 1.

(c) Where (if anywhere) is f discontinuous?

Solution: $x = -2, 1$.

(d) What is $\lim_{x \to -1} f(x)$?

Solution: 1

(e) What is $f(-1)$?

Solution: 1

(f) What is $\lim_{x \to 1} f(x)$?

Solution: This is a jump discontinuity and so the limit doesn't exist.

(g) What is $f(1)$?

Solution: This value also doesn't exist since it's out of the domain.