

$$I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_m A = A = A I_n$$

$$ax = b \Rightarrow x = \frac{1}{a} \cdot b$$

$$A\vec{x} = \vec{b} \Rightarrow \vec{x} = \frac{1}{A} \vec{b} ??$$

What does $\frac{1}{A}$ mean?

Want a B , s.t. $BA = I_n$

$$BA\vec{x} = B\vec{b}$$

$$\text{If } I_n \vec{x}$$

$$\vec{x}$$

Dfn: B is the inverse of A

$$\text{if } BA = AB = I_n \\ B = A^{-1}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/10 & 2/5 \\ 3/10 & -1/5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} \text{ inconsistent!}$$

so $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has no inverse

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 1 & 1 & | & 0 \end{bmatrix} \xrightarrow{\text{RR}} \begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 1 & 1 & | & 0 & 1 \end{bmatrix}$$

$$M_{m \times n} \xrightarrow{\text{R}} A \vec{x} = \vec{b} \quad R^m \quad R^n$$

$$\begin{bmatrix} A & | & I \end{bmatrix} \xrightarrow{\text{RR}} \begin{bmatrix} I & | & A^{-1} \end{bmatrix}$$

$\checkmark \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad]$

not invertible

$$A \vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1} \vec{b}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -6 & -1 & 1 \end{array} \right]$$

not invertible!

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array} \right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

In general $AB \neq BA$

But $AA^{-1} = I_n = A^{-1}A$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Find } \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$$\begin{bmatrix} ax+bx & ay+bw \\ cx+dz & cy+dw \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} ab & 1 & 0 \\ cd & 0 & 1 \end{bmatrix}$$



$$\begin{aligned} ax + bx + cy + dw &= 1 \\ cx + dz + by + bw &= 0 \\ ox + oz + ay + bv &= 0 \\ ox + oz + cy + dw &= 1 \end{aligned}$$

$$\left[\begin{array}{cc|cc|c} a & b & 0 & 0 & 1 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & d & 1 \end{array} \right]$$

Q: When is a matrix invertible?

TFAE (the following are equivalent):

1) A is invertible

2) L is invertible. $L(L^{-1}(\vec{x})) = \vec{x}$ $L^{-1}(L(\vec{x})) = \vec{x}$

3) $\vec{A}\vec{x} = \vec{b}$ has a unique sol for any \vec{b} .

$$\vec{A}\vec{x} = \vec{b} \Rightarrow \vec{x} = \vec{A}^{-1}\vec{b}$$

4) $a_1\vec{v}_1 + \dots + a_n\vec{v}_n = \vec{b}$ has a unique sol for any \vec{b}

5) $N(A)$ is trivial, $\text{columns span } \mathbb{R}^m$.

6) L is 1-1 and onto

7) A is row-equivalent to I_n .

A and B are row-equivalent
if you can reach B from A
w/ 1 row operations.

elementary matrices

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 5 & -1 & 1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 13 & -1 & 6 \\ 5 & -1 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Elementary matrices do row operations
all are invertible

If A is RE to I_n , then

$$A = E_1 E_2 \cdots E_K I_n$$

$$A^{-1} = (E_1 E_2 \cdots E_K)^{-1} = E_K^{-1} \cdots E_2^{-1} E_1^{-1}$$

$$\begin{aligned} ABA^{-1}B^{-1} \\ (A|B)(B^{-1}|A^{-1}) &= A|I_n|A^{-1} \\ &= AA^{-1} = I_n \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$R_1 = R_1 + R_2 \quad R_2 = 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ ? \end{bmatrix}$$



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ ? \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$A\vec{x}A^{-1} = \vec{b}A^{-1}$$

~~$$A\vec{x} = A^{-1}\vec{b}$$~~

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = DNE \quad ?$$

$$A\vec{x} = \vec{b}_1$$

$$A\vec{x} = \vec{b}_2$$

$$A\vec{x} = \vec{b}_3$$

$$A\vec{x} = \vec{b}_{1,2,3}$$

$$A\vec{x} = \vec{b}_{1,2,3,4}$$

$$A\vec{x} = \vec{b}_{1,2,3,5}$$

1) what if A is not invertible?

Can I still precompute?

2) can build every \vec{b} out of
these vectors,