

# Dfn: Vector Space

Set  $V$ ,

vector addition

$$\vec{v} + \vec{w}$$

scalar multiplication

$$r\vec{v}$$

1. (Closure under addition)  $\mathbf{u} + \mathbf{v} \in V$

2. (Additive commutativity)  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

3. (Additive associativity)  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

4. (Additive identity) There is an element  $\mathbf{0} \in V$  called the "zero vector", such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for every  $\mathbf{u}$ .

5. (Additive inverses) For each  $\mathbf{u} \in V$  there is another element  $-\mathbf{u} \in V$  such that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix} = \vec{0}$$

6. (Closure under scalar multiplication)  $r\mathbf{u} \in V$

7. (Distributivity)  $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$

8. (Distributivity)  $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$

9. (Multiplicative associativity)  $r(s\mathbf{u}) = (rs)\mathbf{u}$

10. (Multiplicative Identity)  $1\mathbf{u} = \mathbf{u}$ .

$P(x) = \{ a_0 + a_1x + \dots + a_nx^n \mid n \in \mathbb{N}, a_i \in \mathbb{R} \}$  is a VS.

$$1 \in P(x)$$

$$3 + 7x - 15x^2 \in P(x)$$

$$8 + \pi x^3 \in P(x)$$

$$34x^3 \in P(x)$$

$$(3 + 7x - 15x^2) + (8 + \pi x^3) = 11 + 7x - 15x^2 + \pi x^3$$

$$\vec{0} = 0 + 0x + 0x^2$$

$$(3 + 7x - 15x^2) + (-3 - 7x + 15x^2) = \vec{0}$$

is the space  $\{y_k\} = \{\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots\}$   
of discrete signals

$$\{\dots, x_{-1}, x_0, x_1, \dots\} + \{\dots, y_{-1}, y_0, y_1, \dots\} = \{\dots, x_{-1} + y_{-1}, x_0 + y_0, x_1 + y_1, \dots\}$$

$$r \{\dots, y_{-1}, y_0, y_1, \dots\} = \{\dots, ry_{-1}, ry_0, ry_1, \dots\} \quad \text{is a V.S.}$$

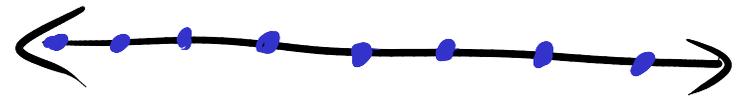
$$\{\dots, 1, 1, 1, 1, \dots\} \quad \{e^{-1/4 t^2}\}$$

$$\{\dots, -1, 1, 2, 3, \dots\}$$

Ex:  $\mathbb{Z}$  the set of integers

$$\begin{array}{ccc} \pi \cdot 3 & \notin & \mathbb{Z} \\ \uparrow & & \uparrow \\ \mathbb{R} & & \mathbb{Z} \end{array}$$

not closed under  
scalar mult  
so not a VS.



$$\begin{array}{ccc} 1.5 \cdot 3 = 4.5 \\ \uparrow & & \uparrow \\ \mathbb{R} & & \mathbb{Z} \end{array}$$

Ex:  $[0, 5] = \{0 \leq x \leq 5 \mid x \in \mathbb{R}\}$

$$\begin{array}{ccc} 4 + 3 = 7 \\ \uparrow & \uparrow & \uparrow \\ [0, 5] & [0, 5] & [0, 5] \end{array}$$

not closed  
under addition  
not a VS

define  $V = \mathbb{R}$

$$x \oplus y = 2x + y$$

not a VS

$$3 \oplus 5 = 2 \cdot 3 + 5 = 11$$

$$5 \oplus 3 = 2 \cdot 5 + 3 = 13$$

Ex:  $M_{m \times n}$  the space of  $m \times n$  matrices

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

$$3 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} \quad \text{Is a VS.}$$

$$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Ex:  $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R}\}$

def  $f+g$  by  $(f+g)(x) = f(x) + g(x)$

$rf$  by  $(rf)(x) = r f(x)$ .

is a VS.

$\vec{0}(x) = 0$   
 $(-f)(x) = -f(x)$

$(f + (-f))(x) = f(x) + (-f(x)) = 0$

so  $(f + (-f)) = \vec{0}$

$\sin(x)$      $e^x$   
 $\cos(x)$      $x^2$

span  $\rightarrow$

$\{a \sin(x) + b \cos(x) + c e^x + d x^2\}$

$$\{ f: \mathbb{R} \rightarrow \mathbb{R} \mid f(0) = 3 \}$$

no  $\vec{0}$  in here, not VS

$$\frac{d}{dx} y = y$$

func,  $n$ -valued eqn

$$\left( \frac{d}{dx} - 1 \right) y = 0$$

what is the kernel  
of  $\left( \frac{d}{dx} - 1 \right)$

$$\ker \left( \frac{d}{dx} - 1 \right) = \{ c e^x \} = \text{span} \{ e^x \}$$

Prop (cancellation): Let  $V$  be a VS

suppose  $\vec{u}, \vec{v}, \vec{w} \in V$

suppose  $\vec{u} + \vec{w} = \vec{v} + \vec{w}$ . Then  $\vec{u} = \vec{v}$ .

$$\left[ \begin{array}{c} \cancel{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cancel{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \end{array} \right]$$

$$\text{PF} / (\vec{u} + \vec{w}) + (-\vec{w}) = (\vec{v} + \vec{w}) + (-\vec{w})$$

$$\text{so } \vec{u} + (\vec{w} + (-\vec{w})) = \vec{v} + (\vec{w} + (-\vec{w}))$$

$$\text{so } \vec{u} + \vec{0} = \vec{v} + \vec{0}$$

$$\text{so } \vec{u} = \vec{v}$$

QED

b/c  $\vec{w}$  must have an additive inverse

add assoc

add inv

add ID.

$$\vec{x} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \vec{y} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

then  $\vec{x} = \vec{y}$

Prop:  $V$  a  $V$ S,  $\vec{u} \in V$ ,  $r \in \mathbb{R}$ . Then

$$1) 0\vec{u} = \vec{0}$$

$$2) r\vec{0} = \vec{0}$$

$$3) (-1)\vec{u} = -\vec{u}$$

Pf/1)  $\vec{u} = 1 \cdot \vec{u} = (0+1)\vec{u}$

$$= 0\vec{u} + 1 \cdot \vec{u}$$

$$= 0\vec{u} + \vec{u}$$

$$\vec{0} + \vec{u} = 0\vec{u} + \vec{u}$$

so  $\vec{0} = 0\vec{u}$

(mult id)

(dist)

(mult id)

(add id)

(cancellation)

3) Want to show that  $\vec{u} + (-1)\vec{u} = \vec{0}$

$$\vec{u} + (-1)\vec{u} = 1\vec{u} + (-1)\vec{u}$$

$$= (1 + (-1))\vec{u} = 0\vec{u}$$

$$= \vec{0}$$

(mult id)

(dist)

(part 1)

$$W = \left\{ \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} \right\}$$

$$\begin{bmatrix} a \\ b \\ a+b \end{bmatrix} + \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = \begin{bmatrix} a+x \\ b+y \\ (a+x) + (b+y) \end{bmatrix} \in W$$

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$$W = \{ c_1 = a+b \}$$

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \end{bmatrix}$$

Know  $c_1 = a_1 + b_1$       so  $c_1 + c_2 = a_1 + a_2 + b_1 + b_2$

$$2x + 2y = 7$$

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$$

$$2a_1 + 2b_1 = 7$$

$$2a_2 + 2b_2 = 7$$

$$2(a_1 + a_2) + 2(b_1 + b_2)$$

$$= 2a_1 + 2b_1 + 2a_2 + 2b_2$$

$$= 7 + 7 = 14$$

not closed under +.