

Column vectors \mathbb{R}^n

polynomials $P(x)$

signals S

functions $\mathcal{F}(R, R)$

Vector Space Subspaces

Def: Let V be a VS

$W \subseteq V$ it is a subspace of V
if W is also a VS.

Prop: V a VS, $W \subseteq V$.

W is a SS iff

- 1) $\vec{0} \in W$
- 2) if $\vec{u}, \vec{v} \in W$, then $\vec{u} + \vec{v} \in W$
- 3) if $r \in R, \vec{u} \in W$, then $r\vec{u} \in W$.

Let $V = P(x) = \{a_0 + a_1 x + \dots + a_n x^n \mid a_i \in \mathbb{R}, n \in \mathbb{N}\}$

Claim $W = \{a_0 + a_1 x\}$ is a SS of V .

$$1) \vec{0} = 0 + 0x \in W$$

$$2) \text{Let } a_0 + a_1 x, b_0 + b_1 x \in W$$

$$\text{then } (a_0 + a_1 x) + (b_0 + b_1 x)$$

$$= (a_0 + b_0) + (a_1 + b_1)x \in W$$

so W is additively closed.

$$3) \text{Let } r \in \mathbb{R}, a_0 + a_1 x \in W.$$

$$\text{Then } r(a_0 + a_1 x) = ra_0 + ra_1 x \in W$$

Thus W is a SS.

More generally,

$$P_n(x) = \{a_0 + a_1 x + \dots + a_n x^n \mid a_i \in \mathbb{R}\}$$

is a SS of $P(x)$

$$U = \{a_0 x^0 + \dots + a_n x^n\} \subseteq V$$

$$1) \vec{0} = 0x^0 + 0x^n$$

$$2) (a_0 x^0 + \dots + a_n x^n) + (b_0 x^0 + \dots + b_n x^n)$$

$$= (a_0 + b_0)x^0 + \dots + (a_n + b_n)x^n \in U$$

$$3) r(a_0 x^0 + \dots + a_n x^n) = ra_0 x^0 + \dots + ra_n x^n \in U$$

So U is a SS

$$W = \{1 + a_i x\}$$

1) Is $\vec{0} \in W$?

2) + closure

3) • closure

1) $\vec{0} \notin W$ b/c $0 \neq 1$

2) $(1 + a_1 x) + (1 + b_1 x) = 2 + (a_1 + b_1)x$ not closed

3) $r(1 + a_1 x) = r + r a_1 x$ not closed.

$$U = \{a_i x\}_{i \in S}$$

$$W = 1 + U$$

$$W = \{a_0 = 1, a_1 = 0, a_2 = 0, \dots\}$$

$$U = \{a_0 = 0, a_1 = 0, \dots\}$$

$V = \mathcal{F}(\mathbb{R}, \mathbb{R})$

$W = D(\mathbb{R}, \mathbb{R}) = \{f \mid f' \text{ exists}\}$
ss of V .

1) Is $\vec{0} \in W$?

A: $f(x) = 0$ is differentiable
so $\vec{0} \in W$.

2) If f', g' exist

$$(f+g)' = f' + g'$$

Closed under +

3) If f' exists,

$$(rf)' = rf' \quad (\text{closed under } \cdot)$$

$C(\mathbb{R}, \mathbb{R})$ cts fns is a ss

$C^\infty(\mathbb{R}, \mathbb{R})$ ∞ diff'able fns

$U = \mathcal{F}(\mathbb{R}, [a, b])$ not a ss

$W = \mathcal{F}([a, b], \mathbb{R})$ ss

U not a ss

$$f(x) = b$$

$$f(x) + f(x) = 2b$$

$$3f(x) = 3b$$

$W = \{f \mid f(x) = f(-x)\} \text{ SS}$
even fns

$V = S$

$W = \text{signals that eventually } 0.$

i.e. $y_K = 0 \vee K > N \text{ SS}$

$U = \text{signals s.t. } y_K = y_{K+4} \text{ SS}$

$X = \{ \{y_K\} \mid y_0 = 0 \} \text{ SS}$
homogeneous

$Y = \{ \{y_K\} \mid y_0 = 1 \} \text{ not a SS}$

1) $\vec{0} \in W$

2) $f(x) = f(-x)$
 $g(x) = g(-x)$

then $(f+g)(x) = f(x)+g(x)$

$$= f(-x)+g(-x) = (f+g)(-x)$$

$f+g \in W$.

3) if $f(x) = f(-x)$

then $r f(x) = r f(-x)$.

Linear Transformations

Dfn: U, V VS

$L: U \rightarrow V$. is a LT, if

$$\bullet L(\bar{u}_1 + \bar{u}_2) = L(\bar{u}_1) + L(\bar{u}_2)$$

$$\bullet L(r\bar{u}) = rL(\bar{u})$$

Dfn: The kernel of L is

$$\{\bar{u} \mid L(\bar{u}) = \vec{0}\}$$

This is a SS of U .
(nullspace)

IF $S \subseteq U$ then

$L(S) = \{L(\bar{u}) \mid \bar{u} \in S\} \subseteq V$.
is the image of S .

If S is a SS, then $L(S)$ is too.

PF/ 1) $\vec{0} \in S$, so $\vec{0} = L(\vec{0}) \in L(S)$.

2) If $\vec{v}_1, \vec{v}_2 \in L(S)$,

then $\exists \bar{u}_1, \bar{u}_2 \in S$ s.t. $L(\bar{u}_1) = \vec{v}_1, L(\bar{u}_2) = \vec{v}_2$.

$$\vec{v}_1 + \vec{v}_2 = L(\bar{u}_1) + L(\bar{u}_2) = L(\bar{u}_1 + \bar{u}_2)$$

$\bar{u}_1 + \bar{u}_2 \in S$ b/c S is a SS

$$\text{so } L(\bar{u}_1 + \bar{u}_2) \in L(S).$$

Df₁: $L(u)$ is the image of L .

This is a SS of V .

(columnspace)

$$V = D([a, b], \mathbb{R})$$

define $\frac{d}{dx} : V \rightarrow \mathcal{F}([a, b], \mathbb{R})$

$$f \mapsto f'$$

$$\ker \frac{d}{dx} = \{f(x) = c\}$$

space of constants.

$\text{im } \frac{d}{dx}$ = integrable functions

$$\frac{d}{dx} (f+g) = \frac{d}{dx} f + \frac{d}{dx} g$$

$$\frac{d}{dx} rf = r \frac{d}{dx} f$$

so $\frac{d}{dx}$ is linear.

def. re $L: V \rightarrow \mathcal{F}$

$$L(f) = f' - f$$

$$\begin{aligned} L(f+g) &= f' + g' - f - g \\ &= L(f) + L(g) \end{aligned}$$

$$L(rf) = rf' - rf = rL(f).$$

linear.

$$\begin{aligned} \ker(L) &= \{f | f' = f\} \\ &= \{e^x\} \end{aligned}$$

exponential growth

$$T(f) = f'' + f$$

$$\begin{aligned} \ker(T) &= \{f | f'' + f = 0\} \\ &= \{f | f'' = -f\} \end{aligned}$$

Simple harmonic motion

$$= \{A \sin(x) + B \cos(x)\}$$

$$I: C([a, b], \mathbb{R}) \rightarrow C([a, b], \mathbb{R})$$

$$F \mapsto \int_a^x f(t) dt$$

$$I(x^2) = \frac{x^3}{3} - \frac{a^3}{3}.$$

This is linear.

$$\begin{aligned} \int_a^x f+g dt &= \int_a^x f dt + \int_a^x g dt \\ \int_a^x rf dt &= r \int_a^x f dt \end{aligned}$$

$$\ker(I) = \{\tilde{0}\}$$

$$\text{im}(I) = \left\{ \text{diff'able fns} \atop \text{s.t. } f(a) = 0 \right\}$$

$$L: C([a, b], \mathbb{R}) \rightarrow \mathbb{R}$$

$$F \mapsto \int_a^b f(t) dt$$

$$\text{linear}$$

$$\ker(L) = \{\text{fns w/ average 0}\}$$

$$\text{im}(L) = \mathbb{R}$$

In quantum, wave $\psi(t)$
evolves by the Schrödinger eqn

$$i\hbar \frac{d}{dt} \psi(t) = \hat{H}(\psi(t))$$

Hamiltonian
a linear operator

ex: $\hat{H} = \frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}, t)$

Want eigenfunctions
eigenvectors of \hat{H}

$$V = S$$

$$R_1 : S \rightarrow S$$

$$R_1(\{e_-, y_-, y_0, y_+, -, z\}) = (\{e_-, y_3, y_-, y_+, -, z\})$$

$$\ker(R_1 - I) = \{ \text{constant signals} \} = \text{span} \{ \{e_1, 1, 1, 1, -, z\} \}$$

$$\ker(R_4 - I) = \{ \text{periodic w/ period 4} \}$$

$$= \text{span} \left\{ \begin{array}{l} \{e_-, 1, 0, 0, 0, 1, 0, 0, 0, -, z\}, \\ \{e_-, 0, 1, 0, 0, 0, 1, 0, 0, -, z\}, \\ \{e_-, 0, 0, 1, 0, 0, 0, 1, 0, -, z\}, \\ \{e_-, 0, 0, 0, 1, 0, 0, 0, 1, -, z\} \end{array} \right\}$$