

Isomorphisms

Defn: $f: U \rightarrow V$ a fn,

if $\exists g: V \rightarrow U$ s.t.

$$f(g(v)) = v, \quad g(f(u)) = u,$$

then $g = f^{-1}$.

If f is an invertible LT,

f is an isomorphism from U to V

and U is isomorphic to V

$$U \cong V.$$

Ex: If E is a basis for U ,

then $[\cdot]_E: U \rightarrow \mathbb{R}^n$ is an iso.

$$L: P_3(x) \rightarrow \mathbb{R}^4$$

$$L(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

is an iso

$$P_3(x) \cong \mathbb{R}^4.$$

Set of polynomials

lists of #s

Set of fns

$$\begin{bmatrix} A \\ B \end{bmatrix}$$

$$\longleftarrow \{ A \sin(x) + B \cos(x) \}$$

Prop: let $L: U \rightarrow V$ be LT
 L is iso iff L is 1-1 and onto.

Pf/: Suppose L is 1-1, onto. Define

$T: V \rightarrow U$ by:

for each $\vec{v} \in V$, there's a unique $\vec{u} \in U$
s.t. $L(\vec{u}) = \vec{v}$. set $T(\vec{v}) = \vec{u}$.

Conversely, suppose L is invertible.

If $L(\vec{u}) = L(\vec{v})$, then

$$\vec{u} = L^{-1}(L(\vec{u})) = L^{-1}(L(\vec{v})) = \vec{v}. \text{ so 1-1}$$

if $\vec{v} \in V$, then $L(L^{-1}(\vec{v})) = \vec{v}$

so $\exists L^{-1}(\vec{v}) \in U$, s.t. $L(L^{-1}(\vec{v})) = \vec{v}$.

so onto.

$D: U \rightarrow \text{span}\{e^x, xe^x, x^2e^x\} \rightarrow \text{span}\{e^x, xe^x, x^2e^x\}$ Is $D: U \rightarrow U$ iso?

$$U \cong \mathbb{R}^3$$

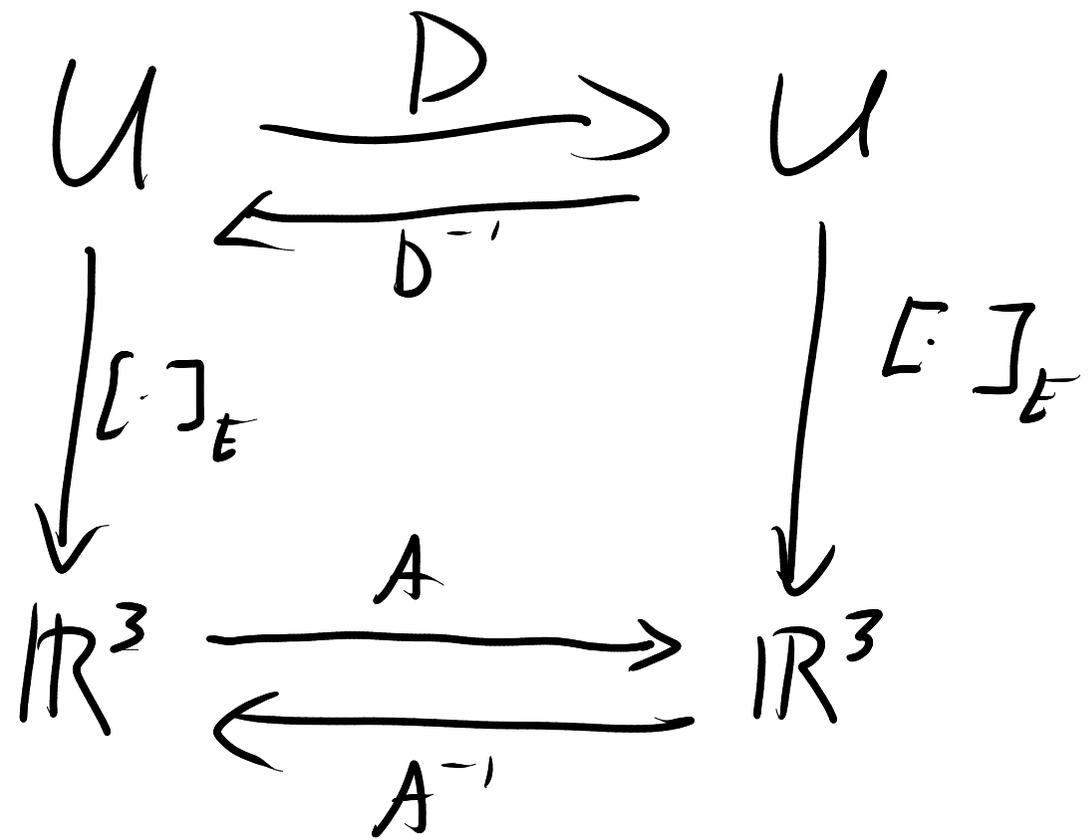
$$ae^x + bxe^x + cx^2e^x \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Find } D^{-1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}. \quad \text{So } D^{-1}(ae^x + bxe^x + cx^2e^x) = (a - b + 2c)e^x + (b - 2c)xe^x + cx^2e^x.$$

$$D^{-1}(D(ae^x + bxe^x + cx^2e^x)) = D^{-1}(\underline{a+b}e^x + \underline{b+2c}xe^x + \underline{c}x^2e^x)$$

$$= (\underline{a+b} - \underline{b+2c} + 2\underline{c})e^x + (\underline{b+2c} - 2\underline{c})xe^x + \underline{c}x^2e^x$$



$$\left[\frac{d}{dx} (2e^x + xe^x + 4x^2e^x) \right]_{\mathcal{E}}$$

||

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Define $E: P_2(x) \rightarrow \mathbb{R}^3$

$$f(x) \mapsto \begin{bmatrix} f(-1) \\ f(0) \\ f(1) \end{bmatrix}$$

$$E(x+x^2) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$E(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$E(x) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$E(x^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array} \right]$$

$$E^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = b + \left(-\frac{1}{2}a + \frac{1}{2}c\right)x + \left(\frac{1}{2}a - b + \frac{1}{2}c\right)x^2$$

Suppose
 $f(-1) = 2$
 $f(0) = 4$
 $f(1) = 6$

i.e. $E(f) = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$f(x) = 4 + 2x$$

I want a parabola through

$$\{1, x, x^2\}$$

$$(0, 4), (3, 7), (7, 1)$$

$$E(f) = \begin{bmatrix} f(0) \\ f(3) \\ f(7) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 9 \\ 1 & 7 & 49 \end{bmatrix}$$

$$A^{-1} = \frac{1}{84} \begin{bmatrix} 84 & 0 & 0 \\ -40 & 49 & -9 \\ 4 & -2 & 3 \end{bmatrix}$$

compute $A^{-1} \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$

$$D: P_3(x) \rightarrow P_3(x)$$

$$f \mapsto f'$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\{1, x, x^2, x^3\}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$D: xP_2(x) \rightarrow P_2(x)$$

$$f \mapsto f'$$

$$\{x, x^2, x^3\}, \{1, x, x^2\}$$

Final comments:

$$1) V \cong V$$

$$\vec{v} \mapsto \vec{v}$$

$$2) L: U \rightarrow V \text{ is an iso}$$

iff sends a basis to a basis

$$3) \text{ if } \dim U = \dim V, \text{ then } U \cong V.$$

Pf/ define $L(a_1 \vec{e}_1 + \dots + a_n \vec{e}_n) = a_1 \vec{f}_1 + \dots + a_n \vec{f}_n$.