

Change of basis  
 vectors in  $E$   $\rightarrow$  vectors in  $F$   
 coords in  $E$   $\leftarrow$  coords in  $F$

Similarity  
 $A \sim B$  if  
 $A = U^{-1}BU$   
 for some  $U$ .

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x+3y+z \\ 2x-y+3z \\ y-z \end{bmatrix}$$

find matrix wrt std basis

$$F = \left\{ \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$B = U^{-1}AU$$

$$= \begin{bmatrix} -1 & -1 & 3 \\ 2 & 2 & -5 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & -16 & -4 \\ 49 & 31 & 7 \\ -38 & -25 & -7 \end{bmatrix}$$

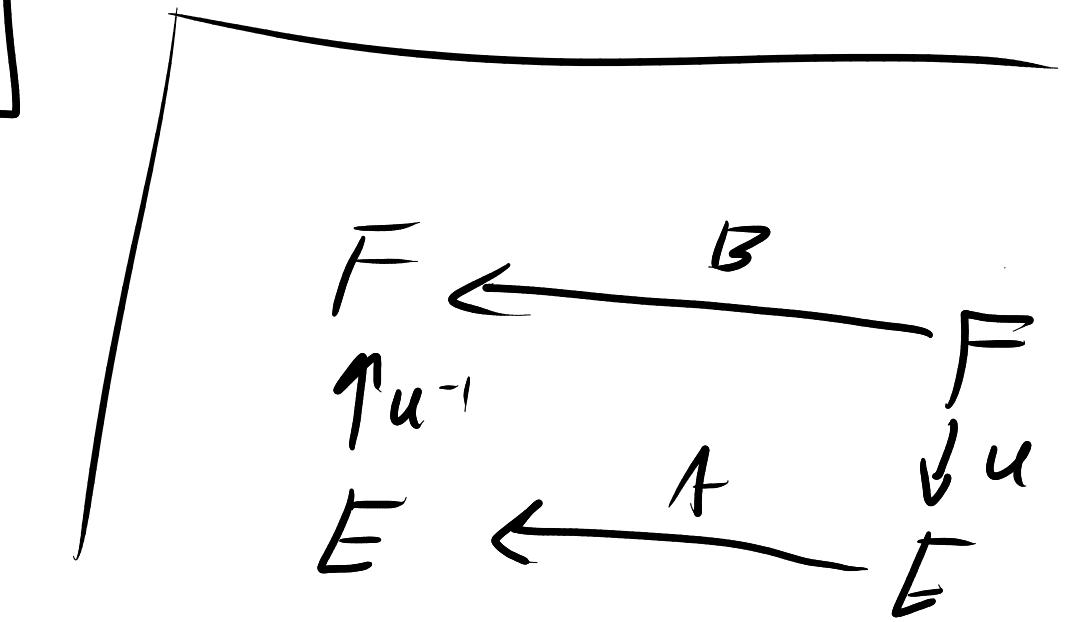
$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 3 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$F \rightarrow E$

$$U^{-1} = \begin{bmatrix} -1 & -1 & 3 \\ 2 & 2 & -5 \\ -1 & -2 & 3 \end{bmatrix}$$

$E \rightarrow F$



$$B = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}$$

evals:  $\lambda = .8 + .6i$   
 $\bar{\lambda} = .8 - .6i$

$$\text{Re}(\vec{v}) = \begin{bmatrix} -4 \\ 10 \end{bmatrix}$$

$$\text{Im}(\vec{v}) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$$U = \begin{bmatrix} -4 & 8 \\ 10 & 0 \end{bmatrix}$$

$$U^{-1} = \frac{1}{80} \begin{bmatrix} 0 & -8 \\ -10 & -4 \end{bmatrix}$$

evals  
 $\begin{bmatrix} -4+8i \\ 10 \end{bmatrix} = \vec{v}$   
 $\begin{bmatrix} -4-8i \\ 10 \end{bmatrix}$

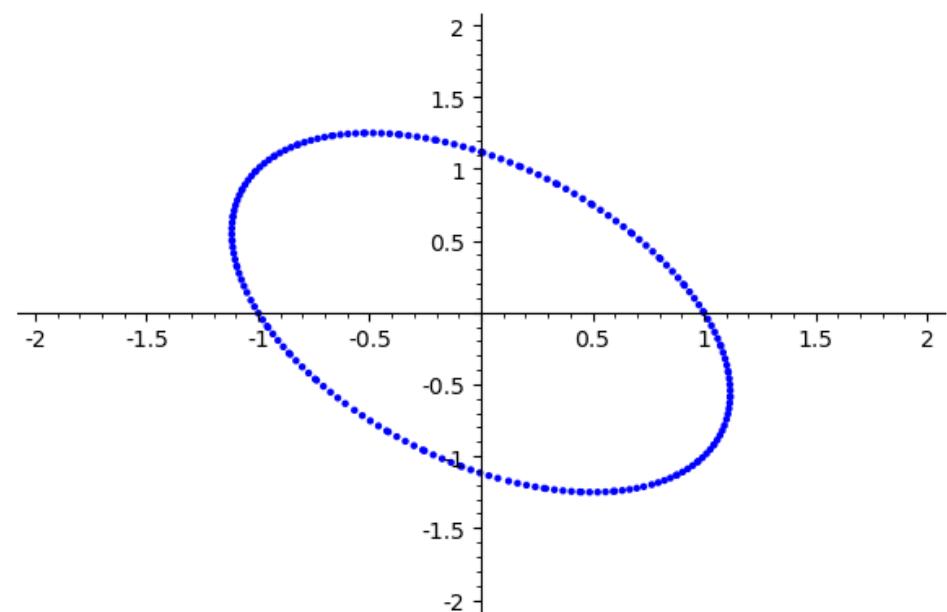
$$C = U^{-1}BU = \frac{1}{80} \begin{bmatrix} 0 & -8 \\ -10 & -4 \end{bmatrix} \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix} \begin{bmatrix} -4 & 8 \\ 10 & 0 \end{bmatrix} = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$$

$$= \cancel{\frac{1}{80} \begin{bmatrix} 0 & -8 \\ -10 & -4 \end{bmatrix}} \begin{pmatrix} -7 & 4 \\ -3 & 6 \end{pmatrix} \cancel{\frac{1}{80} \begin{bmatrix} -4 & 8 \\ 10 & 0 \end{bmatrix}} = \frac{1}{80} \begin{bmatrix} 82 & -48 \\ -48 & 82 \end{bmatrix}$$

This is a rotation matrix!

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \theta = .64$$

$$B = UCU^{-1}$$



Similar matrices are very similar.

If  $A \sim B$ , then:

$$\text{rk}(A) = \text{rk}(B)$$

$$\dim \text{null}(A) = \dim \text{null}(B)$$

$A^{-1}$  exists, if  $B^{-1}$  exists

$$\det(A) = \det(B)$$

$$\chi_A(\lambda) = \chi_B(\lambda)$$

$$\text{evals}(A) = \text{evals}(B).$$

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$$\det(A) = \det(U^{-1}BU)$$

$$= \cancel{\det(U^{-1})} \det(B) \cancel{\det(U)}$$

$$= \det(B).$$

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B = U^{-1}AU = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\chi_A(\lambda) = \det \begin{bmatrix} 2-\lambda & 3 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda)^2 \\ = 2-5\lambda+4\lambda^2-\lambda^3$$

$$\det(A) = 2.$$

$$\chi_B(\lambda) = \det \begin{bmatrix} -\lambda & 2 & 0 \\ 2 & 3-\lambda & 3 \\ 1 & -2 & 1-\lambda \end{bmatrix} = -\lambda(3-\lambda)(1-\lambda) + 6 + 0 \\ = -(0 + 6\lambda + 4 - 4\lambda) \\ = 2-5\lambda+4\lambda^2-\lambda^3$$

Converse is not true!

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\chi_A = (1-\lambda)^2 = \chi_I(\lambda)$$

Is  $A \sim I$ ?

$$A = U^{-1} I U$$

$$= U^{-1} U = I$$

$\Rightarrow \Leftarrow$ .

So  $A \not\sim I$ .

A has 1 even  
1 generic  
I has 2 evens

If A has  $u_1, u_2, u_3$  <sup>E</sup>vecs  
B has  $v_1, v_2, v_3$  <sup>F</sup>vecs

A change of basis is  $F \rightarrow E$

$$B = U^{-1} A U$$

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$$U B = \cancel{U} U^{-1} A U$$

If  $A \sim B$

then  $\chi_A(\lambda) = \chi_B(\lambda)$

so all coeffs are the same.

constant term  $\det$

$$\det(A) = \chi_A(0)$$

$$\det(B) = \chi_B(0)$$

Dfn: the trace of  $A$

is  $(-1)^{n-1} a_{n-1}$  where

$$\chi_A(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_{n-1} \lambda^{n-1} + a_n \lambda^n$$

Prop: • If  $A \sim B$ , then  $\text{tr}(A) = \text{tr}(B)$ .

•  $\text{tr}(A) = \text{sum of eigenvalues}$

$$(\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

$$(-1)^n + (-1)^{n-1}(\lambda_1 + \lambda_2 + \dots + \lambda_n)$$

$$+ (-1)^{n-2}(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \dots + \lambda_{n-1} \lambda_n)$$

•  $\text{tr}(A) = \text{sum of diagonal entries.}$

$$A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 1 \\ 2 & -3 & 2 \end{bmatrix}, \quad \text{tr}(A) = 3 + 4 + 2 = 9.$$

$$B = \begin{bmatrix} 5 & 1 & 1 \\ 4 & -3 & -3 \\ 2 & 1 & 3 \end{bmatrix}, \quad \text{tr}(B) = 5 - 3 + 0 = 2$$

Is  $A \sim B$ ? no

$$A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 1 \\ 2 & -3 & 2 \end{bmatrix} \quad \text{tr}(A) = 9$$

$$C = \begin{bmatrix} 4 & -2 & 3 \\ 5 & 1 & 7 \\ 1 & 1 & 9 \end{bmatrix} \quad \text{tr}(C) = 9$$

$$\chi_A(\lambda) = -\lambda^3 + 9\lambda^2 - 24\lambda + 26$$

$$\chi_C(\lambda) = -\lambda^3 + 9\lambda^2 - 17\lambda - 22$$

not similar!

$\vec{v} \notin \ker(A - \lambda I)$   
 $v \in \ker(A - \lambda I)^n$   
 $\vec{v}$  generalized eigenvector.

if  $v$  is eigenvector of degree 2,  
 then  $A\vec{v} = \lambda v + w$  where  
 $w$  is eigenvector of degree 1.

$$(A - \lambda I) \vec{v} = \vec{w}$$

# Diagonal operators

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

- $\det = d_1 d_2 \cdots d_n$

product of  
diagonal entries

- evals are  $d_1, \dots, d_n$   
w/vecs  $\vec{e}_1, \dots, \vec{e}_n$   
std basis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

definitely  
diagonalizable

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Dfn: A is diagonalizable if  
it is similar to a diagonal matrix.

Prop

- A is diagonalizable iff evals span  $\mathbb{R}^n$   
iff n LI evals

- If n distinct evals then diagonalizable.