

$A \sim B$; if $\exists U$ s.t.

$$U^{-1}AU = B.$$

- same evals
- same det
- same trace

solve $AU = UB$
or
change of basis
btwn evals

Diagonal matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

diag entries are evals

evecs are \vec{e}_i

Cons $A \in M_n$, $F = \{\vec{f}_1, \dots, \vec{f}_n\}$

basis of evecs.

Set U trans matrix $F \rightarrow \text{std}$,
then $U^{-1}AU$ is diagonal.

$$\text{Ex: } A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

evals: 4 $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
-3 $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$F = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$$

$$U = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$V = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$V^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix}$$

$$V^T AV = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$U^{-1}AU = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 & 3 \\ 4 & -9 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 28 & 0 \\ 0 & -21 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}.$$

$$B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \circ \quad \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$U^{-1}BU = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$BU \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = B \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$U^{-1}BU \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = U^{-1} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad (\text{Jordan Canonical form})$$

$$E_2 = \text{Span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$E_4 = \text{Span} \left\{ \begin{bmatrix} 0 \\ i \\ 0 \\ 0 \end{bmatrix} \right\}$$

vecs don't span
not diagonalizable.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & .001 \end{bmatrix}$$

X

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2(-3) & 0 \\ 0 & 0 & 5 \cdot 2 \end{bmatrix}$$

If $A \sim B$, then $A^n \sim B^n$

If $A = U^{-1}BU$, then

$$A^n = (U^{-1}BU)^n = U^{-1}B\cancel{U}\cancel{U^{-1}}\cancel{B}\cancel{U}\cancel{U^{-1}}BU = U^{-1}BU$$

$$= U^{-1}B^nU$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}^{10} = \begin{bmatrix} 3^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 5^{10} \end{bmatrix}$$

$$D = U^{-1} A U$$

$$\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$A = U D U^{-1}$$

$$A^5 = U D^5 U^{-1}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1024 & 0 \\ 0 & -243 \end{bmatrix} \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3072 & 1024 \\ 243 & -486 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 5901 & 2534 \\ 3801 & -434 \end{bmatrix} = \begin{bmatrix} 843 & 362 \\ 543 & -62 \end{bmatrix}$$

Q: What's A^5 ?

$$B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B = UDU^{-1}$$

$$B^n = (UDU^{-1})^n = U D^n U^{-1}$$

$$= U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n U^{-1} = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U^{-1}$$

$$= UDU^{-1} = B.$$

Want formula for B^n .

$B^n = B$ for any n .
("idempotent")

Prop: If A is diagonalizable
and evals are all 0 or 1,
then A is idempotent.

Matrix exponentiation $B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

What is e^B ?

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$e^B = I + B + \frac{1}{2} B^2 + \frac{1}{3!} B^3 + \dots$$

$$= I + U D U^{-1} + \frac{1}{2} U D^2 U^{-1} + \frac{1}{3!} U D^3 U^{-1} + \dots$$

$$= U (I + D + \frac{1}{2} D^2 + \frac{1}{3!} D^3 + \dots) U^{-1}$$

$$= U e^D U^{-1} = U \begin{bmatrix} 1 & e^0 & e^0 \\ 0 & e^1 & 0 \\ 0 & 0 & e^1 \end{bmatrix} U^{-1}$$

$$P = \begin{bmatrix} 0 & e^0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^D = \begin{bmatrix} e^0 & 0 & 0 \\ 0 & e^1 & 0 \\ 0 & 0 & e^1 \end{bmatrix}$$

Markov Chains

every year 6% in suburbs move to the city

2% in city move to suburbs.

$$A = \begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix}$$

$$\begin{bmatrix} s_1 \\ c_1 \end{bmatrix} = A \begin{bmatrix} s_0 \\ c_0 \end{bmatrix}$$

every $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, .92$

Q: What are s_5, c_5
 s_{10}, c_{10}
 What is equilibrium?

$$\begin{bmatrix} s_5 \\ c_5 \end{bmatrix} = A^5 \begin{bmatrix} ? \\ .3 \end{bmatrix} = \begin{bmatrix} .55 \\ .45 \end{bmatrix}.$$

$$A^5 = U D^5 U^{-1} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .92 \end{bmatrix}^5 \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

$$\approx \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .66 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \approx \begin{bmatrix} .744 & .085 \\ .256 & .915 \end{bmatrix}$$

$$\begin{bmatrix} s_{10} \\ c_{10} \end{bmatrix} = A^{10} \begin{bmatrix} ? \\ .3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .92^{10} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .45 \\ .55 \end{bmatrix}.$$

start at $\begin{bmatrix} .70 \\ .30 \end{bmatrix}$

$$A = \begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix}$$

$$\begin{bmatrix} s_1 \\ c_1 \end{bmatrix} = A \begin{bmatrix} s_0 \\ c_0 \end{bmatrix}$$

evens

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix}, 1$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, .92$$

$$\begin{bmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/4 a + 1/4 b \\ 3/4 a + 3/4 b \end{bmatrix}$$

$$= \begin{bmatrix} 1/4(a+b) \\ 3/4(a+b) \end{bmatrix}$$

What is equilibrium?

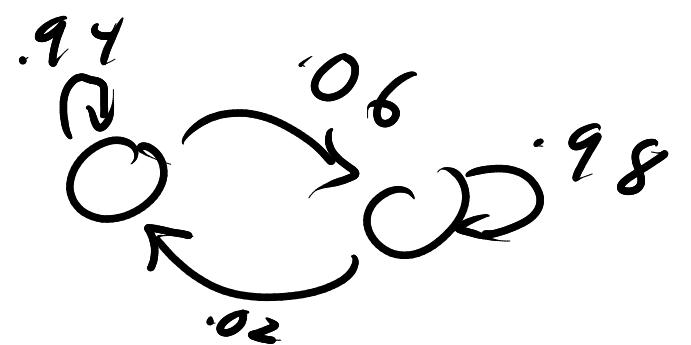
$$\lim_{n \rightarrow \infty} A^n \begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$$

$$\lim_{n \rightarrow \infty} A^n = \lim_{n \rightarrow \infty} U D^n U^{-1}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .92 \end{bmatrix}^n \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \lim_{n \rightarrow \infty} \begin{bmatrix} 1^n & 0 \\ 0 & .92^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{bmatrix}$$



$$C = \begin{bmatrix} .8 & .1 & .05 & .05 \\ .1 & .8 & .05 & .05 \\ .05 & .05 & .8 & .1 \\ .05 & .05 & .1 & .8 \end{bmatrix}$$

Sedan Sport Minivan SUV

evacs

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Sedan Sport Minivan SUV

$$\lim_{n \rightarrow \infty} C^n$$

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} u^{-1} = \begin{bmatrix} .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \end{bmatrix}$$