

# Geometry and Projection

- 1) find coordinates for vectors easily
- 2) reintroduce geometry

Dfn: let  $\vec{u} = (u_1, \dots, u_n), \vec{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$

The dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{i=1}^n u_i v_i$$

$$= \vec{u}^T \vec{v}$$

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = [u_1 v_1 + u_2 v_2 + u_3 v_3]$$

$$\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = 3 \cdot 2 + 1 \cdot (-3) + 4 \cdot 5 \\ = 23.$$

Dfn: let  $\vec{v} \in \mathbb{R}^n$ . The (euclidean or  $L_2$ ) magnitude or norm of  $\vec{v}$  is

$$\|\vec{v}\|_2 = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

distance b/w 2 pts

$$\|\vec{u} - \vec{v}\|$$

Prop: let  $\vec{u}, \vec{v} \in \mathbb{R}^n$ . Then

1) (Positive definite)  $\vec{u} \cdot \vec{u} \geq 0$ ,  
and  $\vec{u} \cdot \vec{u} = 0$  iff  $\vec{u} = \vec{0}$ .

Pf/  $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \dots + u_n^2 \geq 0$   
 $= 0$  iff every term is 0.

2) (Symmetric):  $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

3) (Bilinear)  $L(\vec{x}) = \vec{x} \cdot \vec{v}$  is linear  
and  $T(y) = \vec{u} \cdot \vec{y}$  is linear.

$$\begin{aligned} Pf / L(r\vec{x}) &= (r\vec{x}) \cdot \vec{v} \\ &= rx_1 v_1 + rx_2 v_2 + \dots + rx_n v_n \\ &= r(x_1 v_1 + x_2 v_2 + \dots + x_n v_n) = r(\vec{x} \cdot \vec{v}). \end{aligned}$$

$$\begin{aligned} L(\vec{x} + \vec{y}) &= (x_1 + y_1) v_1 + \dots + (x_n + y_n) v_n \\ &= x_1 v_1 + \dots + x_n v_n + y_1 v_1 + \dots + y_n v_n \\ &= \vec{x} \cdot \vec{v} + \vec{y} \cdot \vec{v}. \end{aligned}$$

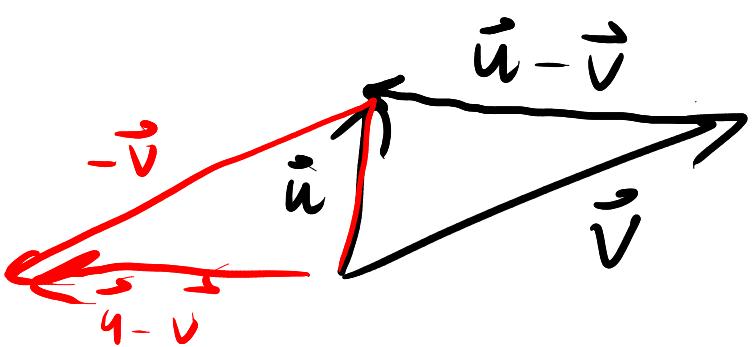
$$\begin{aligned} r(\vec{x} \cdot \vec{v}) &= (r\vec{x}) \cdot \vec{v} \\ (\vec{x} + \vec{y}) \cdot \vec{v} &= \vec{x} \cdot \vec{v} + \vec{y} \cdot \vec{v}. \end{aligned}$$

Prop: If  $\vec{u}, \vec{v} \neq \vec{0} \in \mathbb{R}^n$ ,

< then there is  $\theta$  then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

Pf) there is a  $\Delta$



Law of Cosines

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \|\vec{v}\| \cos \theta$$

$$\begin{aligned} \|\vec{u}\| \cdot \|\vec{v}\| \cos \theta &= \frac{1}{2} (\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2) \\ &= \frac{1}{2} (\vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})) \\ &= \frac{1}{2} (\cancel{\vec{u} \cdot \vec{u}} + \cancel{\vec{v} \cdot \vec{v}} - (\cancel{\vec{u} \cdot \vec{u}} - \cancel{\vec{v} \cdot \vec{u}} - \cancel{\vec{u} \cdot \vec{v}} + \cancel{\vec{v} \cdot \vec{v}})) \\ &= \frac{1}{2} (\vec{v} \cdot \vec{u} + \vec{u} \cdot \vec{v}) = \vec{u} \cdot \vec{v}, \\ \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} \end{aligned}$$

$$\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad < \text{but then?}$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} = \frac{-3 + 28}{\sqrt{9+16} \cdot \sqrt{1+49}}$$

$$= \frac{25}{5\sqrt{50}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

$$\text{So } \theta = \pi/4$$

Sometimes we want a unit vector.

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{(3, 4)}{\sqrt{25}} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}.$$

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{(-1, 2)}{\sqrt{25}} = \begin{bmatrix} -1/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}$$

Thm (Cauchy-Schwarz Inequality)

$\vec{u}, \vec{v} \in \mathbb{R}^n$ . Then

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

This is = iff either

- $r\vec{u} = \vec{v}$  for  $r \in \mathbb{R}$ .
- $\vec{u} = \vec{0}$
- $\vec{v} = \vec{0}$

$$\text{PS: } |\vec{u} \cdot \vec{v}| = \|\vec{u}\| \cdot \|\vec{v}\| |\cos \theta|$$

$$|\cos \theta| \leq 1$$

$$|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \cdot \|\vec{v}\| |\cos \theta| \leq \|\vec{u}\| \cdot \|\vec{v}\|$$

So when are they =?

$$\cancel{\|\vec{u}\| \|\vec{v}\| |\cos \theta|} = \cancel{\|\vec{u}\| \|\vec{v}\|}$$

holds if

- 1)  $\|\vec{u}\| = 0 \Leftrightarrow \vec{u} = \vec{0}$
- 2)  $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = \vec{0}$
- 3)  $|\cos \theta| = 1 \Leftrightarrow \theta = 0 \text{ or } \pi$   
iff  $\vec{u}, \vec{v}$  point in  
same or opposite direction.

best angle: Right angles

if  $\Theta = \pi/2$ , then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos \theta = 0.$$

Dfn:  $\vec{u}$  and  $\vec{v}$  are orthogonal

If  $\vec{u} \cdot \vec{v} = 0$ .

Ex!  $\vec{0}$  is orthogonal to  $\vec{u} \quad \forall \vec{u} \in \mathbb{R}^n$ .

•  $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$  orth to  $\begin{bmatrix} -4 \\ 6 \end{bmatrix}$

$$\left( \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 6 \end{bmatrix} = 3(-4) + 2(6) = 0. \right)$$

Let  $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$  can we find a vector orthogonal to  $\vec{f}$ ?

Solve  $\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \cdot \vec{x} = 0$

$$2x_1 + 3x_2 + 2x_3 = 0.$$

$$\begin{bmatrix} 2 & 3 & 2 \end{bmatrix} \vec{x} = 0$$

set orthogonal to  $\vec{u}$  is

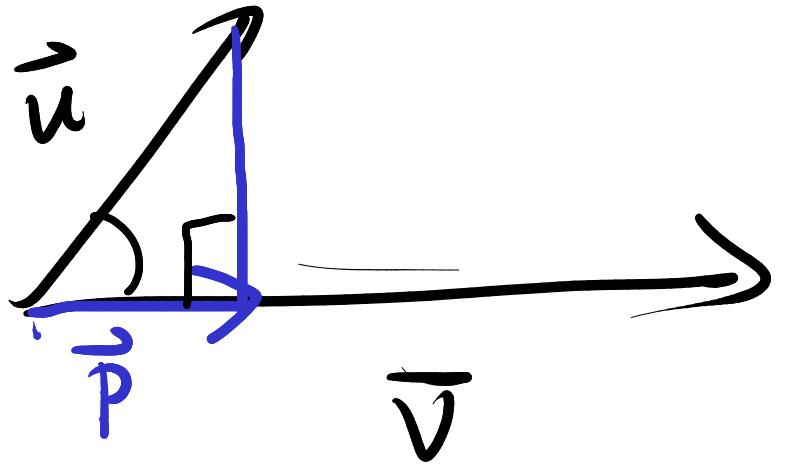
$$N(\vec{u}^\top) = \left\{ \begin{bmatrix} -\alpha - 3\beta \\ 2\beta \\ \alpha \end{bmatrix} \right\}$$

Projection

$\vec{u}, \vec{v}$ . How much of  $\vec{u}$  goes in the direction of  $\vec{v}$ ?

Want  $\|\vec{p}\|$

$$\|\vec{p}\| = \cos \theta \|\vec{u}\|$$



$$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \|\vec{u}\|$$

$$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$$

$$\vec{p} = \|\vec{p}\| \frac{\vec{v}}{\|\vec{v}\|}$$

$$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\| \|\vec{v}\|} \vec{v}$$

$$= \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}.$$

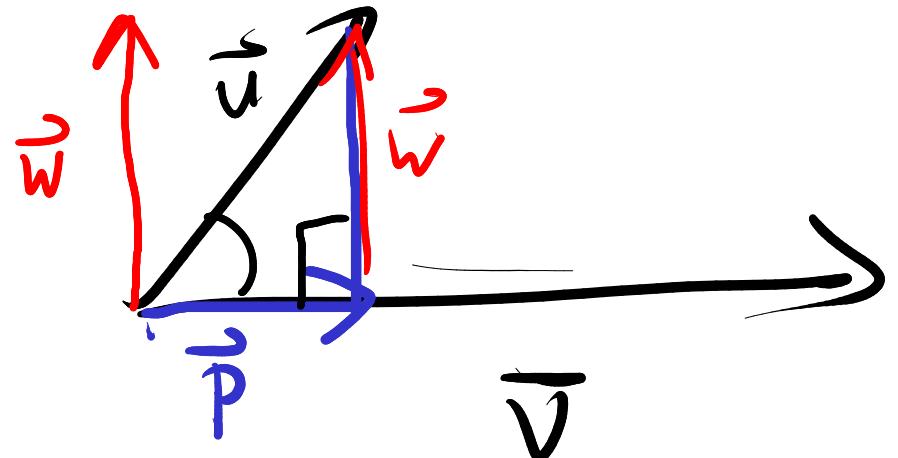
Dfn: The projection of  $\vec{u}$  onto  $\vec{v}$  is

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}.$$

This is a linear function of  $\vec{u}$ .

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$$\text{Can write } \vec{w} = \vec{u} - \vec{p}$$



$$\begin{aligned}\vec{w} \cdot \vec{v} &= (\vec{u} - \vec{p}) \cdot \vec{v} \\ &= \vec{u} \cdot \vec{v} - \vec{p} \cdot \vec{v} \\ &= \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \cdot \vec{v} \\ &= \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} (\vec{v} \cdot \vec{v}) \\ &= \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} = 0,\end{aligned}$$

We have written  $\vec{u} = \vec{p} + \vec{w}$   
where  $\vec{w} \perp \vec{v}$ .  
 $\vec{p}$  same direction as  $\vec{v}$ .

$$\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{3(-1) + 4(7)}{(-1)^2 + 7^2} \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \frac{25}{50} \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 7/2 \end{bmatrix}$$

$$\vec{w} = \vec{u} - \text{proj}_{\vec{v}} \vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 1/2 \end{bmatrix}$$

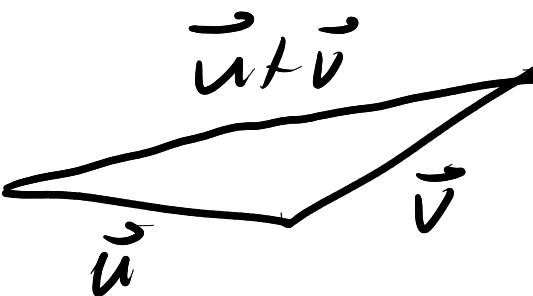
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$$\vec{w} \cdot \vec{v} = \begin{bmatrix} 7/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 7 \end{bmatrix} = -\frac{7}{2} + \frac{7}{2} = 0.$$

Bonus: Norms

way of measuring distance / size e

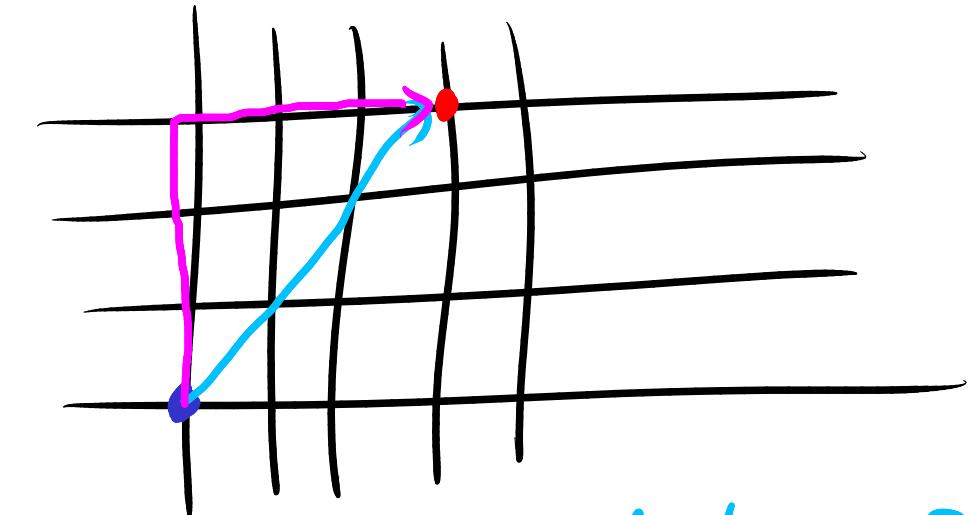
- $\|\vec{v}\| \geq 0$ , and  $\|\vec{v}\|=0$  iff  $\vec{v}=\vec{0}$
- $\|r\vec{v}\| = |r| \|\vec{v}\|$
- (Triangle inequality):  
$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$



Other examples

$$L_2: \|\vec{v}\|_2 = \sqrt{\vec{v} \cdot \vec{v}}$$

$$\|\vec{v}\|_1 = |v_1| + |v_2| + \dots + |v_n|$$



euclidean distance  $3\sqrt{2}$

taxicab distance: 6

$$\|\vec{v}\|_\infty = \max \{ |v_1|, |v_2|, \dots, |v_n| \}$$

