

dot product

b,g, idea: projection

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

Dfn: An inner product
on a vs V is an operator

$$V \times V \rightarrow \mathbb{R}$$

$$\vec{u}, \vec{v} \mapsto \langle \vec{u}, \vec{v} \rangle \text{ s.t. }$$

1) Positive defn, b: $\langle \vec{u}, \vec{u} \rangle \geq 0$,
and $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$.

2) Symmetric: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

3) Bilin corr: $\langle \alpha \vec{u} + \beta \vec{v}, \vec{w} \rangle = \alpha \langle \vec{u}, \vec{w} \rangle + \beta \langle \vec{v}, \vec{w} \rangle$

$$\text{Ex: } V = C([a, b], \mathbb{R})$$

$$\text{define } \langle f, g \rangle = \int_a^b f(t) g(t) dt$$

$$1) \langle f, f \rangle = \int_a^b f(t)^2 dt \geq 0$$

$$\text{if } \langle f, f \rangle = 0$$

$$\int_a^b f(t)^2 dt = 0, \text{ then } f(t) \text{ must be } 0.$$

$$2) \langle f, g \rangle = \int_a^b f(t) g(t) dt \quad \text{are the same}$$

$$\langle g, f \rangle = \int_a^b g(t) f(t) dt$$

$$3) \langle \alpha f + \beta g, h \rangle = \int_a^b (\alpha f(t) + \beta g(t)) h(t) dt$$
$$= \int_a^b \alpha f(t) h(t) + \beta g(t) h(t) dt$$
$$= \alpha \int_a^b f(t) h(t) dt + \beta \int_a^b g(t) h(t) dt = \alpha \langle f, h \rangle + \beta \langle g, h \rangle.$$

ex: $V = P_n(x)$

fix $x_0, \dots, x_n \in \mathbb{R}^n$

define $\langle f, g \rangle = \sum_{i=0}^n f(x_i) g(x_i)$

1) Pos def: $\langle f, f \rangle = \sum_{i=0}^n f(x_i)^2 \geq 0$
b/c $f(x_i)^2 \geq 0$.

If $\langle f, f \rangle = 0$, then $\sum_{i=0}^n f(x_i)^2 = 0$

so $f(x_i) = 0$ for each i .

so f is deg n poly w/ n+1 roots

so $f(x) = 0$

2) Symmetric

$$\langle f, g \rangle = \sum_{i=0}^n f(x_i) g(x_i)$$

$$\langle g, f \rangle = \sum_{i=0}^n g(x_i) f(x_i)$$

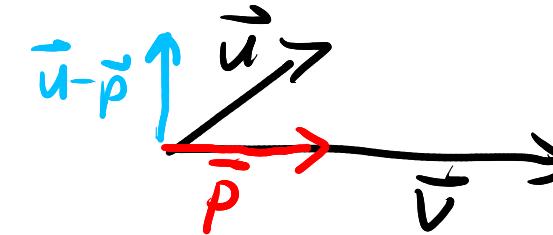
3) Bilinearity

$$\langle \alpha f + \beta g, h \rangle = \sum_{i=0}^n (\alpha f(x_i) + \beta g(x_i)) h(x_i)$$

$$= \alpha \sum_{i=0}^n f(x_i) h(x_i) + \beta \sum_{i=0}^n g(x_i) h(x_i)$$

$$= \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

Dfn: if $\langle \vec{u}, \vec{v} \rangle = 0$,
then \vec{u}, \vec{v} are orthogonal.



Dfn: $\vec{u}, \vec{v} \in V, \vec{v} \neq 0$.

define

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

$$\text{Pf/ 1) } \langle \vec{u} - \vec{p}, \vec{p} \rangle = \langle \vec{u}, \vec{p} \rangle - \langle \vec{p}, \vec{p} \rangle$$

$$\begin{aligned} &= \left\langle \vec{u}, \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} \right\rangle - \left\langle \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}, \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} \right\rangle \\ &= \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \langle \vec{u}, \vec{v} \rangle - \frac{\langle \vec{u}, \vec{v} \rangle^2}{\langle \vec{v}, \vec{v} \rangle^2} \cancel{\langle \vec{v}, \vec{v} \rangle} = 0 \end{aligned}$$

Prop: define $p = \text{proj}_{\vec{v}} \vec{u}$.

$$1) \langle \vec{u} - \vec{p}, \vec{p} \rangle = 0$$

2) if $u = \beta v$, then

$$2) \vec{u} = \vec{p} \text{ iff } \vec{u} = \beta \vec{v}$$

$$p = \frac{\langle u, v \rangle}{\langle v, v \rangle} v = \frac{\langle \beta v, v \rangle}{\langle v, v \rangle} v = \beta \cancel{\frac{\langle v, v \rangle}{\langle v, v \rangle} v} = u.$$

If $u = p$, then $u = \frac{\langle u, v \rangle}{\langle v, v \rangle} v$. So take $\beta = \frac{\langle u, v \rangle}{\langle v, v \rangle}$.

$$V = \mathcal{P}([-1, 1], \mathbb{R})$$

$\forall x \in V$

$$\|1\| = \sqrt{\langle 1, 1 \rangle} = \sqrt{\int_{-1}^1 1 \cdot 1 \, dx}$$
$$= \sqrt{2}$$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\int_{-1}^1 x^2 \, dx}$$
$$= \sqrt{x^3 \Big|_{-1}^1} = \sqrt{2/3}$$

$$\langle 1, x \rangle = \int_{-1}^1 1 \cdot x \, dx = \frac{x^2}{2} \Big|_{-1}^1 = 0$$

$$v = 3 + 2x$$
$$\langle v, 1 \rangle = \int_{-1}^1 3 + 2x \, dx = 3x + x^2 \Big|_{-1}^1 = 6$$
$$\text{proj}_1 v = \frac{\langle v, 1 \rangle}{\langle 1, 1 \rangle} 1 = \frac{6}{2} 1 = 3$$

$$\langle v, x \rangle = \int_{-1}^1 3x + 2x^2 \, dx = \frac{3}{2}x^2 + \frac{2}{3}x^3 \Big|_{-1}^1 = 4/3$$
$$\text{proj}_x \vec{v} = \frac{\langle v, x \rangle}{\langle x, x \rangle} x = \frac{4/3}{2/3} x = 2x.$$

$$\langle x^3, 1 \rangle = \int_{-1}^1 x^3 \, dx = \frac{x^4}{4} \Big|_{-1}^1 = 2/3$$

$$\langle x^3, x \rangle = \int_{-1}^1 x^3 \, dx = \frac{x^4}{4} \Big|_{-1}^1 = 0.$$

Pythagorean Law

If $\|\vec{u}\|, \|\vec{v}\|$ are \perp ,

$$\text{then } \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Pf/ write it out.

Thm (Cauchy-Schwarz inequality)

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$$

Pf/ Let $\vec{p} = \text{proj}_{\vec{v}} \vec{u}$

$$\|\vec{u}\|^2 = \|\vec{p}\|^2 + \|\vec{u} - \vec{p}\|^2$$

$$\|\vec{p}\|^2 = \langle \vec{p}, \vec{p} \rangle = \frac{\langle \vec{u}, \vec{v} \rangle^2}{\langle \vec{v}, \vec{v} \rangle}$$

$$\|\vec{u}\|^2 - \|\vec{u} - \vec{p}\|^2 = \|\vec{p}\|^2 = \frac{\langle \vec{u}, \vec{v} \rangle^2}{\langle \vec{v}, \vec{v} \rangle}$$

$$\|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u} - \vec{p}\|^2 \|\vec{v}\|^2 = \langle \vec{u}, \vec{v} \rangle^2$$

$$\langle \vec{u}, \vec{v} \rangle^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2$$

equal iff $\|\vec{u} - \vec{p}\| = 0$

iff $\vec{u} = \vec{p}$
iff $\vec{u} = \beta \vec{v}$.

$$\text{Cor: } -1 \leq \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \leq 1$$

$$\text{define } \theta = \arccos \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

dot product

showed $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$

\Rightarrow C-S \neq

inner prod

proved C-S \neq

\Rightarrow can define

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

Orthogonal bases

Defn: a set $S = \{\vec{u}_1, \dots, \vec{u}_n\}$

is orthogonal if

$$\langle \vec{u}_i, \vec{u}_j \rangle = 0 \quad \forall i \neq j.$$

Prop: If $S = \{\vec{u}_1, \dots, \vec{u}_n\}$

is orthogonal, and $\vec{u}_i \neq \vec{0}$,

then it is LI.

Pf/ Suppose

$$a_1 \vec{u}_1 + \dots + a_n \vec{u}_n = \vec{0}$$

$$\langle a_1 \vec{u}_1 + \dots + a_n \vec{u}_n, a_1 \vec{u}_1 + \dots + a_n \vec{u}_n \rangle = 0$$

$$0 = \sum_{i,j}^n a_i a_j \langle \vec{u}_i, \vec{u}_j \rangle = \sum_{i=1}^n a_i^2 \underbrace{\langle \vec{u}_i, \vec{u}_i \rangle}_{\geq 0} > 0$$

So each $a_i^2 \langle \vec{u}_i, \vec{u}_i \rangle = 0$

since $\langle \vec{u}_i, \vec{u}_i \rangle \neq 0$, that means $a_i^2 = 0$.

So $a_i = 0$. and the set is LI.