Symmetric Matrices

A is symmetric if $A^T = A$

$$A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

If X is a matrix, then $X^T X$ is symmetric.

$$(X^T X)^T = X^T (X^T)^T = X^T X.$$

$$\chi_A(\lambda) = -(\lambda - 8)(\lambda - 6)(\lambda - 3)$$

$$\vec{v}_8 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_6 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

orthogonal set!

$$\vec{e}_8 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \vec{e}_6 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$D = U^{-1} A U$$

$$U = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$U^{-1} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = U^T$$

Prop: • If $A = A^T$, it has an orth basis of evecs

• If U has orthonormal cols, then $U^{-1} = U^T$.

Pf/ 1) we will prove that evecs are $\perp$.

$\vec{v}_1 \ \lambda_1 \quad \vec{v}_2 \ \lambda_2$.

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T (A\vec{v}_2)$$
$$= \vec{v}_1^T \lambda_2 \vec{v}_2$$
$$= \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)$$

$$(\vec{v}_1^T A)\vec{v}_2 = (\vec{v}_1^T A^T)\vec{v}_2$$
$$= (A\vec{v}_1)^T \vec{v}_2 = \lambda_1 \vec{v}_1^T \vec{v}_2$$
$$= \lambda_1 \vec{v}_1 \cdot \vec{v}_2$$

$$\lambda_2 \vec{v}_1 \cdot \vec{v}_2 = \lambda_1 \vec{v}_1 \cdot \vec{v}_2$$

So either $\lambda_1 = \lambda_2$ or $\vec{v}_1 \cdot \vec{v}_2 = 0$.

2) Suppose $U = [\vec{c}_1 \ \vec{c}_2 \cdots \vec{c}_n]$ has orth columns

claim $U^T U = I$.

$$U^T U = \begin{bmatrix} \vec{c}_1 \\ \vec{c}_2 \\ \vdots \\ \vec{c}_n \end{bmatrix} [\vec{c}_1 \ \vec{c}_2 \cdots \vec{c}_n] = \begin{bmatrix} \vec{c}_1 \cdot \vec{c}_1 & \vec{c}_1 \cdot \vec{c}_2 & \cdots & \vec{c}_1 \cdot \vec{c}_n \\ \vec{c}_2 \cdot \vec{c}_1 & \vec{c}_2 \cdot \vec{c}_2 & \cdots & \vec{c}_2 \cdot \vec{c}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{c}_n \cdot \vec{c}_1 & \vec{c}_n \cdot \vec{c}_2 & \cdots & \vec{c}_n \cdot \vec{c}_n \end{bmatrix}$$

$\|u\| = \sqrt{u \cdot u}$

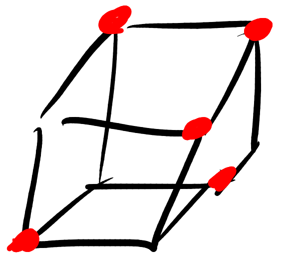
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 2\cdot 4 & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & \cdots & & 0 & 1 \end{bmatrix} = I.$$

Principal Component Analysis

each senator gets a vector of 1s and 0s.

100 vectors in $\mathbb{R}^{600}$



$\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\hat{x}_1 = \begin{bmatrix} -3/5 \\ -3/5 \\ -3/5 \end{bmatrix}, \hat{x}_2 = \begin{bmatrix} 2/5 \\ 2/5 \\ -3/5 \end{bmatrix}, \hat{x}_3 = \begin{bmatrix} 2/5 \\ -3/5 \\ 2/5 \end{bmatrix}, \hat{x}_4 = \begin{bmatrix} 2/5 \\ 2/5 \\ 2/5 \end{bmatrix}, \hat{x}_5 = \begin{bmatrix} -3/5 \\ 2/5 \\ 2/5 \end{bmatrix}$

$M = \begin{bmatrix} 3/5 \\ 3/5 \\ 3/5 \end{bmatrix}$

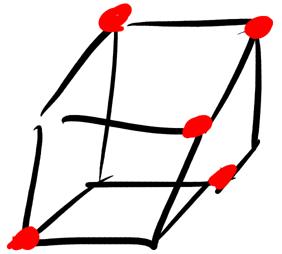
$B = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 & 2 & -3 \\ -3 & 2 & -3 & 2 & 2 \\ -3 & -3 & 2 & 2 & 2 \end{bmatrix}$

Variance: how much vector varies. $\frac{1}{n-1}\vec{v}\cdot\vec{v}$

Covariance: how much 2 vectors change together $\frac{1}{n-1}\vec{u}\cdot\vec{v}$

Covariance matrix $S = \frac{1}{n-1} B B^T = \frac{1}{50}\begin{bmatrix} -3 & 2 & 2 & 2 & -3 \\ -3 & 2 & -3 & 2 & 2 \\ -3 & -3 & 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} -3 & -3 & 3 \\ 2 & 2 & -3 \\ 2 & -3 & 2 \\ 2 & 2 & 2 \\ -3 & 2 & 2 \end{bmatrix}$

$= \frac{1}{50}\begin{bmatrix} 30 & 5 & 5 \\ 5 & 30 & 5 \\ 5 & 5 & 30 \end{bmatrix}.$



$$\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{x}_1 = \begin{bmatrix} -3/5 \\ -3/5 \\ -3/5 \end{bmatrix}, \quad \hat{x}_2 = \begin{bmatrix} 2/5 \\ 2/5 \\ -3/5 \end{bmatrix}, \quad \hat{x}_3 = \begin{bmatrix} 2/5 \\ -3/5 \\ 2/5 \end{bmatrix} \quad \hat{x}_4 = \begin{bmatrix} 2/5 \\ 2/5 \\ 2/5 \end{bmatrix} \quad \hat{x}_5 = \begin{bmatrix} -3/5 \\ 2/5 \\ 2/5 \end{bmatrix}$$

$$S = \frac{1}{50} \begin{bmatrix} 30 & 5 & 5 \\ 5 & 30 & 5 \\ 5 & 5 & 30 \end{bmatrix}.$$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$4/5$ $\qquad$ $1/2$ $\qquad$ $1/2$

$\vec{p}_1 = -\frac{9}{15} \vec{u}_1$

$\vec{p}_2 = \frac{1}{15} \vec{u}_1$

$\vec{p}_3 = \frac{1}{15} \vec{u}_1$

$\vec{p}_4 = \frac{6}{15} u_1$

$\vec{p}_5 = \frac{1}{15} u_1$

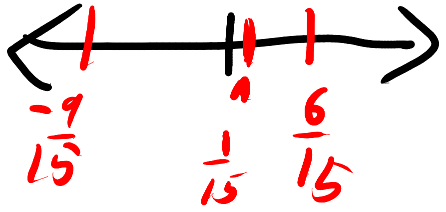
$q_1 = 0 u_2$

$q_2 = 1/2 u_2$

$q_3 = 0 u_2$

$q_4 = 0 u_2$

$q_5 = -1/2 u_2$



ex) IQ tests

ex) house size

ex) bird size

$$\begin{bmatrix} 1 \\ 2 \\ 30 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$