Math 2184 Midterm Solutions

Instructor: Jay Daigle

October 20, 2020

1. This test is due Tuesday at midnight. Logistically, this will work just like the mastery quizzes: download it, write up your answers, and upload them to Blackboard for us to grade.
2. You will have two hours for this test. Please write down your start and end times on the test and include that in your upload. You may not spend more than two hours on the test unless you have a specific accommodation.
3. You are not allowed to consult books or notes during the test, but you may use a one-page cheat sheet you have made for yourself ahead of time. Please upload your sheet along with your test.
4. If you have questions, I will be online and responsive during the usual class times. If you want to take the test at a time you know I'll be able to answer any questions quickly, I encourage you to use one of those time slots.
5. You may use a calculator, but don't use anything that has the ability to do row-reductions for you. Using a calculator for basic arithmetic is fine.
6. The maximum score for this test is 80 points.

Name:

Time Started:

Time Completed:

1	
2	
3	
4	
$\sum$	

Problem 1.

(a) (10 points) Find the set of solutions to the following system of linear equations:

$$\begin{aligned} 2x + 2z &= 6 \\ -3x - 4y + 9z &= -29 \\ x + 2y - 5z &= 13 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 6 \\ -3 & -4 & 9 & -29 \\ 1 & 2 & -5 & 13 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -5 & 13 \\ -3 & -4 & 9 & -29 \\ 2 & 0 & 2 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -5 & 13 \\ 0 & 2 & -6 & 10 \\ 0 & -4 & 12 & -20 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the set of solutions is $\{(3 - \alpha, 5 + 3\alpha, \alpha)\}$.

(b) (10 points) Prove that the following vectors in $\mathcal{P}_2(x)$ are linearly independent, *explicitly using the formal definition of linear independence*:

$$f(x) = -1 + x^2, \quad g(x) = 2 + x - x^2, \quad h(x) = 3x + 2x^2$$

Solution: Suppose we have $af + bg + ch = 0$. Then we have

$$-a + ax^2 + 2b + b - bx^2 + 3cx + 2cx^2 = 0$$

and so

$$(-a + 2b) + (b + 3c) + (a - b + 2c) = 0,$$

which gives us the homogeneous system of linear equations

$$\begin{aligned} -a + 2b &= 0 \\ b + 3c &= 0 \\ a - b + 2c &= 0 \end{aligned}$$

and thus

$$\begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and thus $a = b = c = 0$. So we've shown that if $af + bg + ch = 0$ then $a = b = c = 0$, and thus f, g, h are linearly independent by definition.

Problem 2.

(a) (10 points) Find the inverse of the matrix $\begin{bmatrix} 0 & -1 & 3 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 3 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 0 & 1 \\ 0 & 1 & -3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 2 & 0 & 1 \\ 0 & 1 & -3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 8 & -15 \\ 0 & 1 & 0 & -1 & -3 & 6 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right]$$

so the inverse is

$$\begin{bmatrix} 2 & 8 & -15 \\ -1 & -3 & 6 \\ 0 & -1 & 2 \end{bmatrix}.$$

(b) (5 points) If $B^{-1} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 5 & 2 \\ 7 & 1 & 4 \end{bmatrix}$ find the set of solutions to $B\mathbf{x} = \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}$.

Solution:

$$\mathbf{x} = B^{-1} \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 5 & 2 \\ 7 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 42 \\ 45 \\ 27 \end{bmatrix}$$

(c) (5 points) Compute

$$\begin{bmatrix} 2 & 5 \\ -1 & -3 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 3 & 5 & -2 \\ 4 & 1 & 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & 5 \\ -1 & -3 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 3 & 5 & -2 \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 26 & 15 & 1 \\ -15 & -8 & -1 \\ -4 & 16 & -12 \end{bmatrix}$$

Problem 3 (10 points each)**.**

(a) Prove that $U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 3x + y = z \right\}$ is a subspace of $\mathbb{R}^3$.

Solution: We need to check three things.

(a) $\mathbf{0} \in U$ since $3 \cdot 0 + 0 = 0$.

(b) If (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in U$, then $3x_1 + y_1 = z_1$ and $3x_2 + y_2 = z_2$. Thus $3(x_1 + x_2) + (y_1 + y_2) = (z_1 + z_2)$, so

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \in U$$

and U is thus closed under addition.

(c) If $(x, y, z) \in U$ and $r \in \mathbb{R}$, then $3x + y = z$ so $3rx + ry = rz$ and thus

$$r \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix} \in U$$

so U is closed under scalar multiplication.

Thus by the subspace theorem this is a subspace.

(b) Is $3 + 4x + 6x^2$ in the span of the set $S = \{3 - 5x, 4 - 2x + 3x^2, 1 + x^2\}$?

Solution: We set up the system of equations

$$\left[\begin{array}{ccc|c} 3 & 4 & 1 & 3 \\ -5 & -2 & 0 & 4 \\ 0 & 3 & 1 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

which has a solution. Thus

$$3 + 4x + 6x^2 = -2(3 - 5x) + 3(4 - 2x + 3x^2) - 3(1 + x^2)$$

is in the span of S.

Problem 4 (4 points each).

Let $L : \mathbb{R}^3 \to \mathcal{P}_2(x)$ be given by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a + b + (a+c)x + (2a+b+c)x^2.$

(a) Prove L is a linear transformation.

(b) Find a matrix for L with respect to the standard bases $\{(1,0,0),(0,1,0),(0,0,1)\}$ and $\{1, x, x^2\}$ for each space respectively.

(c) Find a basis for $\ker(L)$.

(d) Find a basis for the image $L(\mathcal{P}_2(x))$.

(e) If L is invertible, find a (non-matrix!) formula for the inverse. If it is not, explain why it is not.

Solution:

(a) (i) If (a_1, b_1, c_1) and $(a_2, b_2, c_2) \in \mathbb{R}^3$, then

$$\begin{aligned} L\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}\right) &= a_1 + b_1 + (a_1 + c_1)x + (2a_1 + b_1 + c_1)x^2 \\ L\left(\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}\right) &= a_2 + b_2 + (a_2 + c_2)x + (2a_2 + b_2 + c_2)x^2 \\ L\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}\right) &= a_1 + a_2 + b_1 + b_2 + (a_1 + c_1 + a_2 + c_2)x + (2a_1 + b_1 + c_1 + 2a_2 + b_2 + c_2)x^2 \\ &= L\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}\right) \end{aligned}$$

(ii) if $(a, b, c) \in \mathbb{R}^3$ and $r \in \mathbb{R}$ then

$$\begin{aligned} L\left(r\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) &= L\left(\begin{bmatrix} ra \\ rb \\ rc \end{bmatrix}\right) = ra + rb + (ra + rc)x + (2ra + rb + rc)x^2 \\ rL\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) &= r\left(a + b + (a+c)x + (2a+b+c)x^2\right) = L\left(r\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right). \end{aligned}$$

Thus L is a linear transformation by definition.

(b) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

(c)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the set of solutions is $\{(-c, c, c)\}$ and has basis $\{(-1, 1, 1)\}$.

(d) The first two columns of the reduced matrix have leading 1s, and thus the first two columns of the original matrix give us a basis for the columnspace. Thus $\{(1,1,2),(1,0,1)\}$ is a basis for the columnspace.

But the function maps into $\mathcal{P}_2(x)$, so we need to interpret these as polynomials. Thus a basis for the actual image of L is $\{1+x+2x^2, 1+x^2\}$.

(e) This function is not invertible. We can justify this in a few different ways: the function has non-trivial kernel and thus is not 1-1; the function has a two-dimensional image and thus is not onto; the associated matrix is not row-equivalent to the identity. There are a few other ways you could justify this claim as well.