Problem 1. (a) Let $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. Find the characteristic polynomial, the eigenvalues, and a basis for each eigenspace.

Solution: $\chi(\lambda) = -\lambda^3 + 4\lambda^2 - 4\lambda$ has roots $2, 2, 0$, with corresponding eigenvectors $(-1, 0, 1), (-1, 1, 0), (1, 0, 1)$.

(b) Find the determinant and trace of $B = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{bmatrix}$.

Solution: $\det(B) = 4$ and $\text{Tr}(B) = 7$. $\det(C) = 0$ and $\text{Tr}(C) = 9$.

Problem 2. (a) Let

$$E = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \text{and} \qquad F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Find the change of basis matrix from E to F.

Solution: The transition matrix from E to the standard basis is

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transition matrix from the standard basis to F is then

$$B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus the transition matrix from E to F is

$$B^{-1}A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -1 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

and the transition matrix from F to E is

$$(B^{-1}A)^{-1} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 2 & 1/2 \\ -3/2 & -5 & -1/2 \end{bmatrix}.$$

(b) Diagonalize the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ and compute A^5.

Solution:

A has eigenvalues $3, 1, 1$ with corresponding eigenvectors $(1,1,1), (-1,0,1), (0,1,0)$. Then we have

$$U = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$U^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

$$D = U^{-1}AU = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^5 = UD^5U^{-1} = U\begin{bmatrix} 243 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}U^{-1} = \begin{bmatrix} 122 & 0 & 121 \\ 121 & 1 & 121 \\ 121 & 0 & 122 \end{bmatrix}.$$

Problem 3.

(a) Let $V = \mathcal{C}([1,3], \mathbb{R})$, with the inner product $\langle f, g \rangle = \int_1^3 f(t)g(t)\,dt$. Find $\|1\|$ and $\|x\|$. Find the projection of $1+x$ onto 1 and onto x.

Solution:

$$\|1\| = \sqrt{\int_1^3 1^2\,dx} = \sqrt{x|_1^3} = \sqrt{2}$$

$$\|x\| = \sqrt{\int_1^3 x^2\,dx} = \sqrt{x^3/3|_1^3} = \sqrt{26/3}$$

$$\text{Proj}_1\, 1+x = \frac{\langle 1+x, 1\rangle}{\langle 1, 1\rangle}1 = \frac{1}{2}\int_1^3 1+x\,dx(1)$$

$$= \frac{1}{2}(x + x^2/2)|_1^3(1) = 3(1)$$

$$\text{Proj}_x\, 1+x = \frac{\langle 1+x, x\rangle}{\langle x, x}x = \frac{3}{26}\int_1^3 x+x^2\,dx(x)$$

$$= \frac{3}{26}(x^2/2 + x^3/3)|_1^3(x) = \frac{3}{26}(4 + 26/3)x = \frac{38}{26}x.$$

(b) Let $U = \text{span}\{(1,1,1,0), (1,0,-1,1)\}$. Find an orthonormal basis for $U^\perp$, and then find the orthogonal decomposition of $(2,-1,5,6)$ with respect U.

Solution:

To find a basis for $U^\perp$, we compute the kernel of

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

which is $\text{span}\{(1,-2,1,0), (-1,1,0,1)\}$. Thus $\{(1,-2,1,0), (-1,1,0,1)\}$ is a basis for $U^\perp$. To get an orthonormal basis, we divide by the norm of each vector, to give

$$\left\{ \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix} \right\}.$$

We see that $(1,1,1,0)\cdot(1,0,-1,1) = 1-1 = 0$, so this is an orthonormal basis for U. We compute

$$\text{Proj}_{(1,1,1,0)}\begin{bmatrix}2\\-1\\5\\6\end{bmatrix} = \frac{(2,-1,5,6)\cdot(1,1,1,0)}{(1,1,1,0)\cdot(1,1,1,0)}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \frac{6}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\2\\2\\0\end{bmatrix}$$

$$\text{Proj}_{(1,0,-1,1)}\begin{bmatrix}2\\-1\\5\\6\end{bmatrix} = \frac{(2,-1,5,6)\cdot(1,0,-1,1)}{(1,0,-1,1)\cdot(1,0,-1,1)}\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} = \frac{3}{3}\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$$

$$\begin{bmatrix}2\\-1\\5\\6\end{bmatrix}_U = \begin{bmatrix}2\\2\\2\\2\end{bmatrix} + \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} = \begin{bmatrix}3\\2\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}2\\-1\\5\\6\end{bmatrix}_{U^\perp} = \begin{bmatrix}2\\-1\\5\\6\end{bmatrix} - \begin{bmatrix}3\\2\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\-3\\4\\5\end{bmatrix}.$$

We can check that the second vector is in fact in $U^\perp$ by taking the inner product with the two basis vectors for U.

Problem 4. Let $A = \begin{bmatrix}1 & -2\\-1 & 2\\0 & 3\\2 & 5\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}3\\1\\-4\\2\end{bmatrix}$.

(a) Find a least-squares solution $\hat{\mathbf{x}}$ to the equation $A\mathbf{x} = \mathbf{b}$.

Solution:

We want to solve

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

$$\begin{bmatrix}1 & -1 & 0 & 2\\-2 & 2 & 3 & 5\end{bmatrix}\begin{bmatrix}1 & -2\\-1 & 2\\0 & 3\\2 & 5\end{bmatrix} = \begin{bmatrix}1 & -1 & 0 & 2\\-2 & 2 & 3 & 5\end{bmatrix}\begin{bmatrix}3\\1\\-4\\2\end{bmatrix}$$

$$\begin{bmatrix}6 & 6\\6 & 42\end{bmatrix} = \begin{bmatrix}6\\-6\end{bmatrix}$$

$$\left[\begin{array}{cc|c}6 & 6 & 6\\6 & 42 & -6\end{array}\right] \to \left[\begin{array}{cc|c}1 & 1 & 1\\0 & 36 & -12\end{array}\right] \to \left[\begin{array}{cc|c}1 & 1 & 1\\0 & 3 & -1\end{array}\right] \to \left[\begin{array}{cc|c}1 & 0 & 4/3\\0 & 1 & -1/3\end{array}\right]$$

so our least-squares solution is $\hat{\mathbf{x}} = (4/3, -1/3)$.

(b) Compute $(\mathbf{b} - A\hat{\mathbf{x}})\cdot A\hat{\mathbf{x}}$. What does this tell you about your solution from part (a)?

Solution:

We compute

$$\begin{aligned}
A\hat{\mathbf{x}} &= \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix} \\
\mathbf{b} - A\hat{\mathbf{x}} &= \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \\
(\mathbf{b} - A\hat{\mathbf{x}}) \cdot A\hat{\mathbf{x}} &= \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix} = 2 - 6 + 3 + 1 = 0.
\end{aligned}$$

This tells us that our least-squares solution $A\hat{\mathbf{x}}$ is orthogonal to the residual error term $\mathbf{b} - A\hat{\mathbf{x}}$. This is because we found a solution to the equation $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the projection of $\mathbf{b}$ onto the image of A. This gives us the closest possible output, and the residual term will always be perpendicular to this image.