Problem 1.

(a) (10 points) Find the set of solutions to the following system of linear equations:

$$\begin{aligned} 3x + 7y + 5z &= 34 \\ 2x + 4y + 2z &= 20 \\ -x + 3z &= -2 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 3 & 7 & 5 & 34 \\ 2 & 4 & 2 & 20 \\ -1 & 0 & 3 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 14 \\ 2 & 4 & 2 & 20 \\ -1 & 0 & 3 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 14 \\ 0 & -2 & -4 & -8 \\ 0 & 3 & 6 & 12 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the set of solutions is $\{(2 + 3\alpha, 4 - 2\alpha, \alpha)\}$.

(b) (10 points) Prove that the following vectors in $\mathcal{P}_2(x)$ are linearly independent, *explicitly using the formal definition of linear independence*:

$$f(x) = 1 + x, g(x) = 3 - x + x^2, h(x) = 2 + 3x^2$$

Solution: Suppose we have $af + bg + ch = 0$. Then we have

$$a + ax + 3b - bx + bx^2 + 2c + 3cx^2 = 0$$

and so

$$(a + 3b + 2c) + (a - b)x + (b + 3c)x^2 = 0$$

which gives us the homogeneous system of linear equations

$$\begin{aligned} a + 3b + 2c &= 0 \\ a - b &= 0 \\ b + 3c &= 0 \end{aligned}$$

and thus

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and thus $a = b = c = 0$. So we've shown that if $af + bg + ch = 0$ then $a = b = c = 0$, and thus f, g, h are linearly independent by definition.

Problem 2.

(a) (10 points) Find the inverse of the matrix $\begin{bmatrix} 2 & -2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 3 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 2 & -2 & 3 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 & -2 \\ 0 & -6 & 1 & 1 & 0 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & -2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & -5 \\ 0 & 1 & 0 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right]$$

so the inverse is

$$\begin{bmatrix} -1 & 4 & -5 \\ 0 & -1/2 & 1 \\ 1 & -3 & 4 \end{bmatrix}.$$

(b) (5 points) If $B^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 6 & 2 & -2 \\ 4 & 1 & 1 \end{bmatrix}$ find the set of solutions to $B\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$.

Solution:

$$\mathbf{x} = B^{-1} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 6 & 2 & -2 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 19 \\ 26 \\ 21 \end{bmatrix}$$

(c) (5 points) Compute

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 4 & 2 \\ -3 & 3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 4 & 2 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 4 \\ 21 & 17 \end{bmatrix}$$

Problem 3 (10 points each)**.**

(a) Prove that $T = \{a_0 + a_1x + a_2x^2 : a_0 = a_1\}$ is a subspace of $\mathcal{P}_2(x)$.

Solution: We need to check three things.

(a) $0 = 0 + 0x + 0x^2$ is an element of the set, since $0 = 0$.

(b) If $a_0 + a_1x + a_2x^2$ and $b_0 + b_1x + b_2x^2$ are elements, then $a_0 = a_1$ and $b_0 = b_1$. Then $a_0 + a_1x + a_2x^2 + b_0 + bx^+b_2x^2 = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$ is an element since $a_0 + b_0 = a_1 + b_1$.

(c) If $a_0 + a_1x + a_2x^2$ is an element and r is a scalar, then $r(a_0 + a_1x + a_2x) = ra_0 + ra_1x + ra_2x^2$ is an element since $ra_0 = ra_1$.

Thus by the subspace theorem this is a subspace.

(b) Is $(3, 2, 5)$ in the span of the set $S = \{(1, 1, 1), (1, 2, 3), (3, 5, 7)\}$?

Solution: We set up the system of equations

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 1 & 2 & 5 & 2 \\ 1 & 3 & 7 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

and since the last equation is $0 = 4$ we have a contradiction. Thus no solution exists to the system of equations, and $(3, 2, 5)$ is not in the span of S.

Problem 4.

Let $T : \mathcal{P}_2(x) \to \mathbb{R}^3$ be given by $T(f) = \begin{bmatrix} f(0) \\ f(2) - f(0) \\ 2f(-2) \end{bmatrix}$.

(a) Prove T is a linear transformation.

(b) Find a matrix for T with respect to the standard bases $\{1, x, x^2\}$ and $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for each space.

(c) Find a basis for $\ker(T)$.

(d) Find a basis for the image $T(\mathcal{P}_2(x))$.

(e) If T is invertible, find a (non-matrix!) formula for the inverse.

Solution:

1. We check the two properties of linear transformations.

   (i)

$$\begin{aligned} T\big(a_0 + a_1 x + a_2 x^2\big) + T\big(b_0 + b_1 x + b_2 x^2\big) &= \begin{bmatrix} a_0 \\ 2a_1 + 4a_2 \\ 2a_0 - 4a_1 + 8a_2 \end{bmatrix} + \begin{bmatrix} b_0 \\ 2b_1 + 4b_2 \\ 2b_0 - 4b_1 + 8b_2 \end{bmatrix} \\ &= \begin{bmatrix} a_0 + b_0 \\ 2(a_1 + b_1) + 4(a_2 + b_2) \\ 2(a_0 + b_0) - 4(a_1 + b_1) + 8(a_2 + b_2) \end{bmatrix} \\ &= T\big((a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2)\big) \end{aligned}$$

   (ii)

$$rT(a_0 + a_1 x + a_2 x^2) = r \begin{bmatrix} a_0 \\ 2a_1 + 4a_2 \\ 2a_0 - 4a_1 + 8a_2 \end{bmatrix} = \begin{bmatrix} ra_0 \\ 2ra_1 + 4ra_2 \\ 2ra_0 - 4ra_1 + 8ra_2 \end{bmatrix} = T\big(r(a_0 + a_1 x + a_2 x^2)\big)$$

   Thus T is a linear transformation by definition of linear transformation.

   Alternatively, we could keep the notation more compressed:

   (i)

$$T(f) + T(g) = \begin{bmatrix} f(0) \\ f(2) - f(0) \\ 2f(-2) \end{bmatrix} + \begin{bmatrix} g(0) \\ g(2) - g(0) \\ 2g(-2) \end{bmatrix} = \begin{bmatrix} f(0) + g(0) \\ f(2) + g(2) - (f(0) + g(0)) \\ 2(f(-2) + g(-2)) \end{bmatrix} = T(f + g)$$

   (ii)

$$rT(f) = r \begin{bmatrix} f(0) \\ f(2) - f(0) \\ 2f(-2) \end{bmatrix} = \begin{bmatrix} rf(0) \\ rf(2) - rf(0) \\ 2rf(-2) \end{bmatrix} = T(rf).$$

   Again, we see that T is a linear transformation by definition.

2. We compute that

$$T(1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}$$

$$T(x^2) = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}$$

   so the matrix of the transformation is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 2 & -4 & 8 \end{bmatrix}$.

3.

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 4\\ 2 & -4 & 8\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & -1 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

So the kernel is trivial and has basis $\{\}$.

4. The map is onto. We can look at the three columns of the original matrix and say that we have basis $\{(1,0,2),(0,2,-4),(0,4,8)\}$; but we could also note that the image is all of $\mathbb{R}^3$ and take the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$.

(In an earlier version of these solutions, I said that we needed to convert to polynomials, but that was an error on my part: the codomain is $\mathbb{R}^3$ so the image should be vectors in $\mathbb{R}^3$.)

5. To invert the matrix we can row-reduce again. We get

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 2 & 4 & 0 & 1 & 0\\ 2 & -4 & 8 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 2 & 0 & 1/2 & 0\\ 0 & -1 & 2 & -1/2 & 0 & 1/4\end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 2 & 0 & 1/2 & 0\\ 0 & 0 & 4 & -1/2 & 1/2 & 1/4\end{array}\right] \to \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1/4 & 1/4 & -1/8\\ 0 & 0 & 1 & -1/8 & 1/8 & 1/16\end{array}\right]$$

so the inverse matrix is $\begin{bmatrix}1 & 0 & 0\\ 1/4 & 1/4 & -1/8\\ -1/8 & 1/8 & 1/16\end{bmatrix}$. This means the inverse function is

$$T^{-1}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = a + \frac{1}{8}(2a+2b-c)x + \frac{1}{16}(-2a+2b+c)x^2.$$