Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 10
Due Midnight on Thursday, November 12

This week's mastery quiz has twelve topics. **Do not answer all questions.** Please answer the problems on the new topic, labeled 17. You may answer *two* of the other topics if you did not get a "mastery" grade on them already. If you are retrying a topic you should complete the entire page.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

17. Similarity and Trace
16. Change of Basis
15. Complex and Generalized Eigenvectors
14. Characteristic Polynomials and Finding Eigensystems
13. Eigenvectors and Determinants
12. Matrices of Linear Transformations
11. Bases and Coordinates
9. Vector Spaces and Subspaces
8. Basis and Dimension
7. Subspaces
6. Matrix Inverses

17. **Similarity and Trace**

(a) Let $A = \begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. Show that A is similar to B.

Solution: You can find that the eigenvectors of B are $(1,0)$ and $(0,1)$, and the eigenvectors of A are $(1,2)$ and $(2,3)$, so we want the transition matrix from the basis for A to the (standard) basis for B, which is $U = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. So we can compute that

$$U^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$
$$U^{-1}AU = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = B.$$

Thus we have $B = U^{-1}AU$ as expected, and so $A \sim B$.

Alternatively we can use brute force, and try to solve $UB = AU$ for a matrix of unknowns $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we have

$$UB = AU$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\begin{bmatrix} -a & 0 \\ -c & 0 \end{bmatrix} = \begin{bmatrix} 3a-2c & 3b-2d \\ 6a-4c & 6b-4d \end{bmatrix}$$
$$\begin{bmatrix} 4 & 0 & -2 & 0 \\ 0 & 3 & 0 & -2 \\ 6 & 0 & -3 & 0 \\ 0 & 6 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

gives $2a = c$ and $3b = 2d$, so one solution is $U = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Again, we compute that $U^{-1}AU = B$, so $A \sim B$.

(b) Let $C = \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$. Show that neither matrix is similar to A.

Solution: We can compute that $\mathrm{Tr}(A) = -1$ and $\mathrm{Tr}(C) = 12$, so they can't be similar. A and D have the same trace, but $\det(A) = 0$ and $\det(D) = -8$, so they can't be similar.

16. **Change of Basis**

Let

$$E = \left\{ \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \right\}, \qquad F = \left\{ \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right\}.$$

Find the transition matrix from E to F.

Solution:

It's easiest to compute the transition matrices to the standard basis and combine. We have

$$A = \begin{bmatrix} 4 & 2 & 4 \\ 3 & -1 & 2 \\ 5 & 4 & 3 \end{bmatrix}$$

as the transition matrix from E to the standard basis. We have

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 0 & 4 \\ 2 & 3 & 2 \end{bmatrix}$$

as the transition matrix from F to the standard basis. So we need to compute B^{-1}:

$$\left[\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 5 & 0 & 4 & 0 & 1 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -10 & -1 & -5 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 1 & -3 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 15 & 1 & -10 \end{array}\right]$$
$$\to \left[\begin{array}{cccccc} 1 & 0 & 0 & 12 & 1 & -8 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -15 & -1 & 10 \end{array}\right]$$

So we have

$$B^{-1} = \begin{bmatrix} 12 & 1 & -8 \\ 2 & 0 & -1 \\ -15 & -1 & 10 \end{bmatrix}$$

and the transition matrix from E to F is

$$B^{-1}A = \begin{bmatrix} 12 & 1 & -8 \\ 2 & 0 & -1 \\ -15 & -1 & 10 \end{bmatrix} \begin{bmatrix} 4 & 2 & 4 \\ 3 & -1 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 11 & -9 & 26 \\ 3 & 0 & 5 \\ -13 & 11 & -32 \end{bmatrix}.$$

15. **Complex and Generalized Eigenvectors**

(a) Let $A = \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix}$. Find all (complex!) eigenvalues of A and find a (possibly complex) eigenvector for each eigenvalue.

Solution: We have

$$\chi_A(\lambda) = \det \begin{bmatrix} 2-\lambda & -4 \\ 4 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 + 16 = \lambda^2 - 4\lambda + 20$$

and by the quadratic formula we have

$$\lambda = \frac{4 \pm \sqrt{16-80}}{2} = 2 \pm \frac{1}{2}\sqrt{-64} = 2 \pm 4i.$$

Set $\lambda = 2 + 4i$, and we have

$$A - \lambda I = \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \to \begin{bmatrix} 4i & 4 \\ 0 & 0 \end{bmatrix}$$

and thus an eigenvector would be $(i, 1)$. By conjugation, we know that an eigenvector for $\overline{\lambda}$ would be $(-i, 1)$.

(b) Let $B = \begin{bmatrix} -5 & -7 & -4 \\ -1 & -3 & -1 \\ 7 & 13 & 6 \end{bmatrix}$. Find an eigenvector of eigenvalue -1, and then find a generalized eigenvector of eigenvalue -1.

Solution:

We have

$$B + I = \begin{bmatrix} -4 & -7 & -4 \\ -1 & -2 & -1 \\ 7 & 13 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$E_{-1} = \ker(B + I) = \{(\alpha, 0, -\alpha)\}.$$

So $(1, 0, -1)$ is an eigenvector of eigenvalue 2.

$$(B + I)^2 = \begin{bmatrix} -5 & -10 & -5 \\ -1 & -2 & -1 \\ 8 & 16 & 8 \end{bmatrix}^2 \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has kernel $\{(-2\alpha - \beta, \alpha, \beta)\}$. So the easy choices here are $(-2, 1, 0)$ or $(-1, 0, 1)$, but there are lots of options. It just can't be in the span of $(1, 0, 1)$.

14. **Characteristic Polynomials and Finding Eigensystems**

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Find the characteristic polynomial, the eigenvalues, and a basis for each associated eigenspace.

Solution:

$$\chi_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 2 & 0 \\ -1 & -1 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{bmatrix}$$
$$= (1 - \lambda)^2(-1 - \lambda) - (1 - \lambda) + 2(1 - \lambda) = (1 - \lambda)(\lambda^2 - 1 + 1) = \lambda^2(1 - \lambda)$$

has roots $0, 0, 1$ So we need to find an eigenvector for each eigenvalue.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$E_0 = \operatorname{span}\{(2, -1, 1)\}$$
$$A - I = \begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$E_1 = \operatorname{span}\{(1, 0, 1)\}$$

13. **Eigenvectors and Determinants**

(a) Let $A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & 4 & 2 \\ 3 & 0 & 4 \end{bmatrix}$. Show that $(1,1,1)$ is an eigenvector of A. What is the corresponding eigenvalue?

Solution:

$$\begin{bmatrix} -1 & 3 & 5 \\ 1 & 4 & 2 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Thus $(1,1,1)$ is an eigenvector with eigenvalue 7.

(b) Compute $\det \begin{bmatrix} 4 & 1 & 2 \\ 3 & 1 & -1 \\ 5 & 2 & 2 \end{bmatrix}$ and $\det \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 2 & 6 & 2 \\ 0 & 0 & 4 & 0 \\ 2 & 5 & 1 & 4 \end{bmatrix}$

Solution:

$$\det \begin{bmatrix} 3 & 5 & 4 \\ 2 & 1 & 3 \\ 1 & 2 & 7 \end{bmatrix} = 21 + 15 + 16 - 4 - 18 - 70 = -40$$

$$\det \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 0 & 3 & 4 \\ 3 & 0 & 0 & 5 \\ 0 & 2 & 3 & 1 \end{bmatrix} = 2(-1)^{4+2} \det \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$
$$= 2\big(45 + 24 + 0 - 0 - 0 - 10\big) = 118.$$

12. **Matrices of Linear Transformations**

(a) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(x,y,z) = (3x - z, 4x + 3y - 2z, x + y - 3z)$. Give a matrix for L with respect to the bases to $E = \{(1,2,3), (1,-2,-3), (1,1,1)$ and $F = \{(0,1,1), (1,0,1), (1,1,0)\}$.

Solution:

To find the matrix, we compute

$$\begin{aligned} L(1,2,3) &= (0,4,-6) \to (-1,-5,5) \\ L(1,-2,-3) &= (6,4,8) \to (3,5,1) \\ L(1,1,1) &= (2,5,-3) \to (0,-3,5) \\ A &= \begin{bmatrix} -1 & 3 & 0 \\ -5 & 5 & -3 \\ 5 & 1 & 5 \end{bmatrix}. \end{aligned}$$

(b) Let $T : \mathbb{R}^4 \to \mathcal{P}_2(x)$ be defined by $T(a,b,c,d) = (a+b) + (b+c)x + (c+d)x^2$. Find a matrix for T with respect to the standard basis for $\mathbb{R}^4$ and the basis $F = \{1, x, x^2\}$.

Solution: To find the matrix we compute

$$\begin{aligned}
T(1,0,0,0) &= 1 \to (1,0,0)\\
T(0,1,0,0) &= 1+x \to (1,1,0)\\
T(0,0,1,0) &= x+x^2 \to (0,1,1)\\
T(0,0,0,1) &= x^2 \to (0,0,1)\\
A &= \begin{bmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}.
\end{aligned}$$

11. **Bases and Coordinates**

(a) Prove that $\{5-x^2, 3+x, 2x-3x^2\}$ is a basis for $\mathcal{P}_2(x)$. (Please explicitly use the formal definition of a basis.)

Solution: Suppose $a(5-x^2)+b(3+x)+c(2x-3x^2) = 0$. Then we get the system of equations

$$\begin{aligned}
5a+3b &= 0\\
b+2c &= 0\\
-a-3c &= 0
\end{aligned}$$

which gives the matrix

$$\begin{bmatrix} 5 & 3 & 0\\ 0 & 1 & 2\\ -1 & 0 & -3\end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3\\ 0 & 1 & 2\\ 0 & 0 & -21\end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

and thus $a=b=c=0$. So by definition of linear independence, the three vectors are independent. Since $\mathcal{P}_2(x)$ has dimension 3, they must also span and thus be a basis.

If you want to check spanning directly, you would instead observe that if we want to solve $a(5-x^2)+b(3+x)+c(2x-3x^2) = \alpha+\beta x+\gamma x^2$ then there will always be a solution, since the unaugmented matrix reduces to the identity.

(b) Let $B = \left\{ \begin{bmatrix} 4\\ -2\\ 1\end{bmatrix}, \begin{bmatrix} 3\\ 0\\ 4\end{bmatrix}, \begin{bmatrix} 1\\ -1\\ 0\end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$. Find $\begin{bmatrix} 5\\ -8\\ -9\end{bmatrix}_B$.

Solution:

We want to express the vector $(7,-4,15)$ as a linear combination of vectors in B. So we want to solve

$$a\begin{bmatrix} 4\\ -2\\ 1\end{bmatrix} + b\begin{bmatrix} 3\\ 0\\ 4\end{bmatrix}, +c\begin{bmatrix} 1\\ -1\\ 0\end{bmatrix} = \begin{bmatrix} 5\\ -8\\ -9\end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 3 & 1 & 5\\ -2 & 0 & -1 & -8\\ 1 & 4 & 0 & -9\end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 0 & -9\\ 0 & 8 & -1 & -26\\ 0 & -13 & 1 & 41\end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 0 & -9\\ 0 & -5 & 0 & 15\\ 0 & -13 & 1 & 41\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 0 & 0 & 0 & 3\\ 0 & 1 & 0 & -3\\ 0 & 0 & 1 & 2\end{array}\right]$$

so the coordinates are

$$\begin{bmatrix} 5 \\ -8 \\ -9 \end{bmatrix}_B = \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}.$$

9. **Vector Spaces and Subspaces**

   (a) Let $V = \mathcal{C}(\mathbb{R}, \mathbb{R})$ be the space of continuous real-valued functions, and $U = \{f : f(0) < 0\}$. Is U a subspace of V? Prove your answer.

   Solution:

   (b) Let $V = \mathcal{C}(\mathbb{R}, \mathbb{R})$ be the space of continuous real-valued functions, and $U = \{f : f(0) = 0\}$. Is U a subspace of V? Prove your answer.

   Solution:

8. **Basis and Dimension** Let $A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 1 & 4 & 3 \\ 2 & 1 & 1 & 2 \end{bmatrix}$.

   (a) Find a basis for the nullspace of A.

   (b) Find a basis for the columnspace of A.

   (c) Find a basis for the rowspace of A.

   (d) What is the rank of A?

   Solution: The reduced row echelon form is $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so

   - The nullspace is $\{3z - w, 5z, z, w\}$ and so has basis $\{(-3, 5, 1, 0), (-1, 0, 0, 1\}$.
   - The columnspace has basis $\{(1, 3, 2), (0, 1, 1)\}$.
   - The rowspace has basis $\{(1, 0, 3, 1), (0, 1, -5, 0)\}$.
   - The rank is 2.

7. **Subspaces** For each of the following sets, determine whether it is a subspace of $\mathbb{R}^3$, and justify your answer using the definition of subspace.

   (a) $U = \{(x, y, z) : 3x + y = 2\}$

   (b) $V = \{(x, y, z) : x^2 - 5y = z\}$

   (c) $W = \{(x, y, z) : x + 2y + 3z = 0\}$.

   Solution:

   (a) $3 \cdot 0 + 0 = 0 \neq 2$ so $\mathbf{0} \notin U$. So U is not a subspace.

   (b) $(1, 0, 1) \in V$ since $1^2 - 5 \cdot 0 = 1$ but $(2, 0, 2) \notin V$ since $2^2 - 5 \cdot 0 \neq 2$. So V is not a subspace.

   (c) W is a subspace. We need to check three things.

i. $0 + 2 \cdot 0 + 3 \cdot 0 = 0$ so $\mathbf{0} \in U$.

Thus by definition, W is a subspace of $\mathbb{R}^3$.

6. **Matrix Inverses**

(a) Find the inverse of $A = \begin{bmatrix} 5 & -1 & 3 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$.

Solution: $A^{-1} = \begin{bmatrix} 4 & -7 & 9 \\ 1 & -2 & 3 \\ -6 & 11 & -14 \end{bmatrix}$.

(b) Using part (a), solve the equations $A\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ and $A\mathbf{y} = \begin{bmatrix} 5 \\ 0 \\ 8 \end{bmatrix}$.

Solution:

$$\mathbf{x} = A^{-1} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 41 \\ 13 \\ -63 \end{bmatrix} \qquad \mathbf{y} = A^{-1} \begin{bmatrix} 5 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 92 \\ 29 \\ -142 \end{bmatrix}.$$