Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 11
Due Midnight on Friday, November 20

This week's mastery quiz has seven topics. **Do not answer all questions.** Please answer the problems on the new topics, labeled 19 and 18. You may answer *one* of the other topics if you did not get a "mastery" grade on them already. If you are retrying a topic you should complete the entire page.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

19. Dot Product and Projection
18. Diagonalization
17. Similarity and Trace
16. Change of Basis
15. Complex and Generalized Eigenvectors
7. Subspaces
4. Linear Transformations

19. **Dot Product and Projection** Let $\mathbf{u} = (1, 4, 2)$ and $\mathbf{v} = (-3, 7, 1)$.

(a) Compute $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

(b) Compute $\text{Proj}_{\mathbf{v}} \mathbf{u}$ and the projection of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

(c) What is the angle between $\mathbf{u}$ and $\mathbf{v}$?

(d) Find a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Solution:

(a) $\|\mathbf{u}\| = \sqrt{1 + 16 + 4} = \sqrt{21}$ and $\|\mathbf{v}\| = \sqrt{9 + 49 + 1} = \sqrt{59}$.

(b)

$$\text{Proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{27}{59} \mathbf{v} = \frac{27}{59} \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}$$

$$u - \text{Proj}_{\mathbf{v}} \mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} - \frac{27}{59} \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 140/59 \\ 47/59 \\ 91/59 \end{bmatrix}$$

(c) $\cos\theta = \frac{27}{\sqrt{21 \cdot 59}}$, so $\theta \approx .69655$.

(d) We want to simultaneously solve $\mathbf{u} \cdot \mathbf{x} = 0$ and $\mathbf{v} \cdot \mathbf{x} = 0$. That gives us the two equations $x + 4y + 2z = 0$ and $-3x + 7y + z = 0$. Converting into a matrix gives us

$$\begin{bmatrix} 1 & 4 & 2 \\ -3 & 7 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 0 & 19 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 10/19 \\ 0 & 19 & 7 \end{bmatrix} \to \begin{bmatrix} 19 & 0 & 10 \\ 0 & 19 & 7 \end{bmatrix}$$

and thus we can take $\mathbf{x} = \begin{bmatrix} -10/19 \\ -7/19 \\ 1 \end{bmatrix}$ as a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

18. **Diagonalization** Let $A = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix}$. Diagonalize A, and use this diagonalization to compute A^{100}.

Solution: We can compute that A has eigenvalues $-1, 0, 1$, with eigenvectors $(1, 2, 0), (1, 1, 1), (1, 2, 2$

respectively. Then we can take

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$U^{-1} = \begin{bmatrix} -1 & 1/2 & 1/2 \\ 2 & -1 & 0 \\ 0 & 1/2 & -1/2 \end{bmatrix}$$

$$\begin{aligned} D = U^{-1}AU &= \begin{bmatrix} -1 & 1/2 & 1/2 \\ 2 & -1 & 0 \\ 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

Then we can compute that

$$\begin{aligned} A &= UDU^{-1} \\ A^{10} = UD^{10}U^{-1} &= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1/2 & 1/2 \\ 2 & -1 & 0 \\ 0 & 1/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1/2 & 1/2 \\ 2 & -1 & 0 \\ 0 & 1/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix}. \end{aligned}$$

17. **Similarity and Trace**

(a) Let $A = \begin{bmatrix} 8 & 10 \\ -3 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 3 \\ -4 & -1 \end{bmatrix}$. Show that A is similar to B.

Solution: You can find that the eigenvectors of B are $(1, -1)$ with value 3, and $(3, -4)$ with value 2. And the eigenvectors of A are $(2, -1)$ with value 3, and $(5, -3)$ with value 2, so we want the transition matrix from the eigenbasis for A to the eigenbasis for B. We can compute that the transition matrix from A to the standard matrix is

$$Q = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$$

and the transition matrix from B to the standard matrix is

$$R = \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}.$$

This means that Q takes standard basis vectors to eigenvectors of A, and R takes standard basis vectors to eigenvectors of B. If we want to send eigenvectors of A to eigenvectors of B, we must use the matrix

$$U = RQ^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$$

So we can compute that

$$\begin{aligned} U^{-1} &= \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \\ U^{-1}BU &= \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 8 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ -3 & -3 \end{bmatrix} = A. \end{aligned}$$

Thus we have $A = U^{-1}BU$ as expected, and so $A \sim B$.

Alternatively we can use brute force, and try to solve $UB = AU$ for a matrix of unknowns $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we have

$$\begin{aligned} UB &= AU \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 6 & 3 \\ -4 & -1 \end{bmatrix} &= \begin{bmatrix} 8 & 10 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \begin{bmatrix} 6a-4b & 3a-b \\ 6c-4d & 3c-d \end{bmatrix} &= \begin{bmatrix} 8a+10c & 8b+10d \\ -3a-3c & -3b-3d \end{bmatrix} \\ \begin{bmatrix} -2 & -4 & -10 & 0 \\ 3 & -9 & 0 & -10 \\ 3 & 0 & 9 & -4 \\ 0 & 3 & 3 & 2 \end{bmatrix} &\to \begin{bmatrix} 3 & 0 & 9 & -4 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Takcing $c = 1, d = 0$ this gives us $a = -3, b = -1$ and thus

$$U = \begin{bmatrix} -3 & -1 \\ 1 & 0 \end{bmatrix}$$

and we can check that indeed, $B = U^{-1}AU$.

(Note: if we take $c = -1, d = 0$, then we get the answer we found earlier.)

(b) Let $C = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 2 \\ 1 & -6 \end{bmatrix}$. Show that neither matrix is similar to A.

Solution: We can compute that $\mathrm{Tr}(A) = 4$ and $\mathrm{Tr}(D) = -3$, so they can't be similar. A and C have the same trace, but $\det(A) = 6$ and $\det(C) = -2$, so they can't be similar.

16. **Change of Basis**

Let

$$E = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \right\}, \qquad F = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Find the transition matrix from E to F.

Solution:

It's easiest to compute the transition matrices to the standard basis and combine. We have

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

as the transition matrix from E to the standard basis. We have

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

as the transition matrix from F to the standard basis. So we need to compute B^{-1}:

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 3 & 5 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & -5 & 0 & -3 & 1 & 6 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccccc} 1 & 0 & 0 & -1/5 & 2/5 & -3/5 \\ 0 & 1 & 0 & 3/5 & -1/5 & -6/5 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

So we have

$$B^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 2 & -3 \\ 3 & -1 & -6 \\ 0 & 0 & 5 \end{bmatrix}$$

and the transition matrix from E to F is

$$B^{-1}A = \frac{1}{5}\begin{bmatrix} -1 & 2 & -3 \\ 3 & -1 & -6 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 2 & 4 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & -8 & -12 \\ -1 & -1 & -4 \\ 5 & 10 & 15 \end{bmatrix}.$$

15. **Complex and Generalized Eigenvectors**

(a) Let $A = \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}$. Find all (complex!) eigenvalues of A and find a (possibly complex) eigenvector for each eigenvalue.

Solution: We have

$$\chi_A(\lambda) = \det\begin{bmatrix} 2-\lambda & -2 \\ 1 & 4-\lambda \end{bmatrix} = (2-\lambda)(4-\lambda) + 2 = \lambda^2 - 6\lambda + 10$$

and by the quadratic formula we have

$$\lambda = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm \frac{1}{2}\sqrt{-4} = 3 \pm i.$$

Set $\lambda = 3 + i$, and we have

$$A - \lambda I = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} \to \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$$

and thus an eigenvector would be $(i-1, 1)$. By conjugation, we know that an eigenvector for $\overline{\lambda}$ would be $(-i-1, 1)$.

(b) Let $B = \begin{bmatrix} 4 & 4 & 4 \\ -11 & -8 & -7 \\ 10 & 8 & 7 \end{bmatrix}$. Find an eigenvector of eigenvalue 2, and then find a generalized eigenvector of eigenvalue 2.

Solution:

We have

$$B - 2I = \begin{bmatrix} 2 & 4 & 4 \\ -11 & -10 & -7 \\ 10 & 8 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 \\ 0 & 12 & 15 \\ 0 & -12 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$E_2 = \ker(B - 2I) = \{(\alpha, -5/2\alpha, 2\alpha)\}.$$

So $(1, -5/2, 2)$ is an eigenvector of eigenvalue 2.

$$(B - 2I)^2 = \begin{bmatrix} 2 & 4 & 4 \\ -11 & -10 & -7 \\ 10 & 8 & 5 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 18 & 0 & -9 \\ -18 & 0 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has kernel $\{(\beta, \alpha, 2\beta)\}$. So the easy choices here are $(0, 1, 0)$ or $(1, 0, 2)$, but there are lots of options. It just can't be in the span of $(1, -5/2, 2)$.

7. **Subspaces** For each of the following sets, determine whether it is a subspace of $\mathbb{R}^3$, and justify your answer using the definition of subspace.

(a) $U = \{(x, y, z) : 3x + 5y + z = 0\}$

(b) $V = \{(x, y, z) : 3x - 2z + 4 = y\}$

(c) $W = \{(x, y, z) : xyz = xy + xz\}$.

Solution:

(a) U is a subspace. We need to check three things.

i. $3 \cdot 0 + 5 \cdot 0 + \cdot 0 = 0$ so $\mathbf{0} \in U$.

ii. Suppose $(x, y, z), (a, b, c) \in U$. Then $3x + 5y + z = 0$ and $3a + 5b + c = 0$, so $3(x + a) + 5(b + y) + (z + c) = 0$ and thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$$

is in U.

iii. Suppose $(x, y, z) \in U$, and $r \in \mathbb{R}$. Then $3x + 5y + z = 0$ so $r3x + 5ryrz = 0$, and so

$$r \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}$$

is in U.

(b) $3 \cdot 0 - 2 \cdot 0 + 4 = 4 \neq 0$ so $\mathbf{0} \notin U$. So V is not a subspace.

(c) $(1,0,0) \in W$ since $0 = 0+0$, and $(0,1,1) \in W$, but $(1,0,0)+(0,1,1) = (1,1,1) \notin W$ since $1 \neq 1+1$. So W is not a subspace of $\mathbb{R}^3$.

4. **Linear Transformations**

(a) Prove directly from the definition that the following function is linear:

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y \\ 3x + z \\ y + z \end{bmatrix}$$

Solution: We can compute that

$$\begin{aligned} L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) &= \begin{bmatrix} x_1 - y_1 \\ 3x_1 + z_1 \\ y_1 + z_1 \end{bmatrix} + \begin{bmatrix} x_2 - y_2 \\ 3x_2 + z_2 \\ y_2 + z_2 \end{bmatrix} \\ &= \begin{bmatrix} (x_1 + x_2) - (y_1 + y_2) \\ 3(x_1 + x_2) + (z_1 + z_2) \\ (y_1 + y_2) + (z_1 + z_2) \end{bmatrix} = L\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}\right) \end{aligned}$$

and

$$cL\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - y \\ 3x + z \\ y + z \end{bmatrix} = \begin{bmatrix} cx - cy \\ 3cx + cz \\ cy + cz \end{bmatrix} = L\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right).$$

(b) Write a matrix for the linear function.

Solution: $\begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(c) Is this function one-to-one, onto, both, or neither? Why?

Solution: Row reducing this matrix gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Because there is no non-zero row, there is always a solution to $A\mathbf{x} = \mathbf{b}$, and thus the function is onto. Because there is no free variable, the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution, so the function is one-to-one.