Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 12
Due Midnight on Thursday, December 3

This week's mastery quiz has seven topics. **Do not answer all questions.** Please answer the problems on the new topics, labeled 19 and 18. You may answer *one* of the other topics if you did not get a "mastery" grade on them already. If you are retrying a topic you should complete the entire page.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

21. Orthogonal Decomposition
20. Inner Products
19. Dot Product and Projection
18. Diagonalization
17. Similarity and Trace
11. Bases and Coordinates

21. **Orthogonal Decomposition** Let $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(a) Find an orthogonal basis for U.

(b) Find a basis for $U^\perp$.

(c) Let $\mathbf{y} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Find the orthogonal decomposition of $\mathbf{y}$ with respect to U.

Solution:

(a) This basis isn't quite orthogonal, so we can use Gram-Schmidt. We take $\mathbf{u}_1 = (1, 1, 0, 1)$, and then we have

$$\text{Proj}_{\mathbf{u}_1} \begin{bmatrix} 1 \\ 5 \\ 1 \\ 0 \end{bmatrix} = \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$$

so our orthogonal basis is $\{(1, 1, 0, 1), (-1, 3, 1, -2)\}$.

(b) To find a basis for $U^\perp$ we take a matrix with rows a basis for U and find the kernel. So we have

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 3 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 4 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/4 & 5/4 \\ 0 & 4 & 1 & -1 \end{bmatrix}$$

so our basis can be $\{(1, -1, 4, 0), (-5, 1, 0, 4)\}$.

(c) We need to have an orthogonal basis for either U or $U^\perp$. We already have one for

U, so we can just compute

$$\text{Proj}_{\mathbf{u}_1} \mathbf{y} = \frac{6}{3}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\2\\0\\2\end{bmatrix}$$

$$\text{Proj}_{\mathbf{u}_2} \mathbf{y} = \frac{-3}{15}\begin{bmatrix}-1\\3\\1\\-2\end{bmatrix} = \begin{bmatrix}1/5\\-3/5\\-1/5\\2/5\end{bmatrix}$$

$$\text{Proj}_U \mathbf{y} = \begin{bmatrix}2\\2\\0\\2\end{bmatrix} + \begin{bmatrix}1/5\\-3/5\\-1/5\\2/5\end{bmatrix} = \begin{bmatrix}11/5\\7/5\\-1/5\\12/5\end{bmatrix}$$

$$\text{Proj}_{U^\perp} \mathbf{y} = \mathbf{y} - \text{Proj}_U \mathbf{y} = \begin{bmatrix}4\\1\\0\\1\end{bmatrix} - \begin{bmatrix}11/5\\7/5\\-1/5\\12/5\end{bmatrix} = \begin{bmatrix}9/4\\-2/5\\1/5\\-7/5\end{bmatrix}.$$

20. **Inner Products**

Let $V = \mathcal{P}_2(x)$ and define $\langle f, g\rangle = 2f(0)g(0) + f(1)g(1) + 3f(2)g(2)$.

(a) Prove that this is an inner product. (You may use the fact that a quadratic polynomial with three roots must be zero.)

(b) Let $f(x) = 1 + x$ and $g(x) = x^2 - 3$. Compute $\|f\|$ and $\|g\|$.

(c) Let $f(x) = 1 + x$ and $g(x) = x^2 - 3$. Compute $\text{Proj}_f g$ and the projection of g orthogonal to f.

Solution:

(a) We have to check three things.

i. (Positive definite) Let f be a polynomial. Then

$$\langle f, f\rangle = 2f(0)f(0) + f(1)f(1) + 3f(2)f(2) = 2f(0)^2 + f(1)^2 + 3f(2)^2 \geq 0$$

since any square is non-negative. Further, if $\langle f, f\rangle = 0$, then we know that $2f(0)^2 + f(1)^2 + 3f(2)^2 = 0$ is a sum of squares. So it's a sum of non-negative terms, which means that $f(0), f(1), f(2)$ must all be zero. Thus f is a quadratic polynomial with three roots, which means it must be zero.

ii. (Symmetry) We compute

$$\langle f, g\rangle = 2f(0)g(0) + f(1)g(1) + 3f(2)g(2) = 2g(0)f(0) + g(1)f(1) + 3g(2)f(2) = \langle g, f\rangle.$$

iii. (Bilinearity) By symmetry, we just need to check one side. So

$$\begin{aligned}\langle \alpha f + \beta g, h\rangle &= 2(\alpha f(0) + \beta g(0))h(0) + (\alpha f(1) + \beta g(1))h(1) + 3(\alpha f(2) + \beta g(2))h(2) \\ &= \alpha(2f(0)h(0) + f(1)h(1) + 3f(2)h(2)) + \beta(2g(0)h(0) + g(1)h(1) + 3g(2)h(2) \\ &= \alpha\langle f, h\rangle + \beta\langle g, h\rangle\end{aligned}$$

(b)

$$\|f\| = \sqrt{2+4+27} = \sqrt{33}$$
$$\|g\| = \sqrt{18+4+3} = \sqrt{25} = 5.$$

(c)

$$\begin{aligned}\operatorname{Proj}_f g &= \frac{\langle f, g\rangle}{\langle f, f\rangle} f \\ &= \frac{-6-4+9}{2+4+27}(1+x) = \frac{-1}{33}(1+x) \\ g - \operatorname{Proj}_f g &= x^2 - \frac{1}{33}x - \frac{100}{33}1\end{aligned}$$

19. **Dot Product and Projection** Let $\mathbf{u} = (2, 6, 3)$ and $\mathbf{v} = (4, 8, 1)$.

(a) Compute $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

(b) Compute $\operatorname{Proj}_{\mathbf{v}} \mathbf{u}$ and the projection of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

(c) What is the angle between $\mathbf{u}$ and $\mathbf{v}$?

(d) Find a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Solution:

(a) $\|\mathbf{u}\| = \sqrt{4+36+9} = \sqrt{49} = 7$ and $\|\mathbf{v}\| = \sqrt{16+64+1} = \sqrt{81} = 9$.

(b)

$$\operatorname{Proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{59}{81}\mathbf{v} = \frac{59}{81}\begin{bmatrix}4\\8\\1\end{bmatrix}$$
$$\mathbf{u} - \operatorname{Proj}_{\mathbf{v}} \mathbf{u} = \begin{bmatrix}2\\6\\3\end{bmatrix} - \frac{59}{81}\begin{bmatrix}4\\8\\1\end{bmatrix} = \begin{bmatrix}-74/81\\14/81\\184/81\end{bmatrix}$$

(c) $\cos\theta = \frac{59}{63}$, so $\theta \approx .358$.

(d) We want to simultaneously solve $\mathbf{u}\cdot\mathbf{x} = 0$ and $\mathbf{v}\cdot\mathbf{x} = 0$. That gives us the two equations $2x + 6y + 3z = 0$ and $4x + 8y + z = 0$. Converting into a matrix gives us

$$\begin{bmatrix}2 & 6 & 3\\4 & 8 & 1\end{bmatrix} \to \begin{bmatrix}2 & 6 & 3\\0 & -4 & -5\end{bmatrix} \to \begin{bmatrix}2 & 0 & -9/2\\0 & 1 & 5/4\end{bmatrix} \to \begin{bmatrix}1 & 0 & -9/4\\0 & 1 & 5/4\end{bmatrix}$$

and thus we can take $\mathbf{x} = \begin{bmatrix}9/4 & -5/4 & 1\end{bmatrix}$ as a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

18. **Diagonalization** Let $A = \begin{bmatrix}2 & 0 & 3\\4 & 0 & 6\\-2 & 1 & -1\end{bmatrix}$. Diagonalize A, and use this diagonalization to compute A^{10}.

Solution: We can compute that A has eigenvalues $2, -1, 0$, with eigenvectors $(1,2,0), (3,4,-2), (1,2$
respectively. Then we can take

$$U = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$U^{-1} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1 & -1/2 & 0 \\ -2 & 1 & -1 \end{bmatrix}$$

$$\begin{aligned} D = U^{-1}AU &= \begin{bmatrix} 0 & 1/2 & 1 \\ 1 & -1/2 & 0 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ 4 & 0 & 6 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

Then we can compute that

$$\begin{aligned} A &= UDU^{-1} \\ A^{10} = UD^{10}U^{-1} &= \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1024 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1 \\ 1 & -1/2 & 0 \\ -2 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1024 & 0 & 1 \\ 2048 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1 \\ 1 & -1/2 & 0 \\ -2 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 513 & 1025 \\ -4 & 1026 & 2046 \\ 2 & -1 & 1 \end{bmatrix}. \end{aligned}$$

17. **Similarity and Trace**

(a) Let $A = \begin{bmatrix} 7 & 12 \\ -4 & -7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. Show that A is similar to B.

Solution: You can find that the eigenvectors of B are $(1,0)$ with value 1, and $(1,-1)$ with value -1. And the eigenvectors of A are $(2,-1)$ with value 1, and $(3,-2)$ with value -1, so we want the transition matrix from the eigenbasis for A to the eigenbasis for B. We can compute that the transition matrix from A to the standard matrix is

$$Q = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$$

and the transition matrix from B to the standard matrix is

$$R = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

We can think of this a couple ways. One is that $Q^{-1}AQ = D = R^{-1}BR$, so we have $A = QR^{-1}BRQ^{-1}$. We can check that this works.

If we don't want to use diagonalization, we can instead note that Q takes standard basis vectors to eigenvectors of A, and R takes standard basis vectors to eigenvectors of B. If we want to send eigenvectors of A to eigenvectors of B, we must use the matrix

$$U = RQ^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

So we can compute that

$$\begin{aligned} U^{-1} &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ U^{-1}BU &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ -4 & -7 \end{bmatrix} = A. \end{aligned}$$

Thus we have $A = U^{-1}BU$ as expected, and so $A \sim B$.

Alternatively we can use brute force, and try to solve $UB = AU$ for a matrix of unknowns $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we have

$$\begin{gathered} UB = AU \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 7 & 12 \\ -4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \begin{bmatrix} 7a - 4b & 12a - 7b \\ 7c - 4d & 12c - 7d \end{bmatrix} = \begin{bmatrix} a + 2c & b + 2d \\ -c & -d \end{bmatrix} \\ \begin{bmatrix} 6 & -4 & -2 & 0 \\ 12 & -8 & 0 & -2 \\ 0 & 0 & 8 & -4 \\ 0 & 0 & 12 & -6 \end{bmatrix} \to \begin{bmatrix} 3 & -2 & -1 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{gathered}$$

Takcing $b = 0, d = 2$ this gives us $a = 1/3, c = 1$ and thus

$$U = \begin{bmatrix} 1/3 & 0 \\ 1 & 2 \end{bmatrix}$$

and we can check that indeed, $A = U^{-1}BU$.

(b) Let $C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 3 \\ 2 & -4 \end{bmatrix}$. Show that neither matrix is similar to A.

Solution: We can compute that $\mathrm{Tr}(A) = 0$ and $\mathrm{Tr}(C) = 5$, so they can't be similar. A and D have the same trace, but $\det(A) = 1$ and $\det(C) = -22$, so they can't be similar.

11. **Bases and Coordinates**

(a) Prove that $\{x^2 + x + 1, 3x - 5, 4 + 3x^2\}$ is a basis for $\mathcal{P}_2(x)$. (Please explicitly use the formal definition of a basis.)

Solution:

(b) Let $B = \left\{ \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 5 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$. Find $\begin{bmatrix} 2 & 0 & -2 \end{bmatrix}_B$.

Solution:

We want to express the vector $(2, 0, -2)$ as a linear combination of vectors in B. So we want to solve

$$a \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + b \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, + c \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 2 \\ 1 & 3 & 0 & 0 \\ 4 & -1 & 5 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

so the coordinates are

$$\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}_B = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}.$$